Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 6

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) Which mathematical structures enter the definition of a causal fermion system? Try to get familiar with these objects.

(ii) What is the idea behind the "principle of the fermionic projector"?

(iii) What is a discrete spacetime? Starting from systems of wave functions in a discrete spacetime, how does one get to the setting of causal fermion systems?

(iv) Why is the closed chain important? Why is the trace of a power of the closed chain real? Why is not positive in general?

(v) Try to follow the arguments why $p = 1$ is the only sensible choice for the causal action.

Exercises

Exercise 6.1: A causal fermion system on ℓ_2

let $\mathcal{H} = \ell_2$ the Hilbert space of square-summable complex-valued sequences, equipped with the scalar product

$$\langle u|v\rangle = \sum_{i=1}^{\infty} \bar{u}_i \, v_i, \quad u = (u_i)_{i\in\mathbb{N}}, \; v = (v_i)_{i\in\mathbb{N}}.$$

For any $k \in \mathbb{N}$, let $x_k \in \mathcal{L}(\mathcal{H})$ the operator defined by

$$(x_k u)_k := u_{k+1}, \quad (x_k u)_{k+1} := u_k, \quad (x_k u)_i = 0 \text{ for } i \notin \{k, k+1\}.$$

In other words,

$$x_k u = \big(\underbrace{0, \ldots, 0}_{k\,-\,1 \text{ entries}}, u_{k+1}, u_k, 0, \ldots \big)$$

Finally, let μ the counting measure on $\mathbb{N}$ (i.e. $\mu(X) = |X|$ equals the cardinality of $X \subset \mathbb{N}$.)

(i) Show that every operator x_k has rank two, is symmetric, and has one positive and one negative eigenvalue. Make yourself familiar with the concept that every operator is a point in $\mathscr{F}$ for spin dimension $n = 1$.

(ii) Let $F : \mathbb{N} \to \mathscr{F}$ be the mapping which to every k associates the corresponding operator x_k. Show that the push-forward measure $\rho = F_* \mu$ defined by $\rho(\Omega) := \mu(F^{-1}(\Omega))$ defines a measure on $\mathscr{F}$. Show that this measure can also be characterized by

$$\rho(\Omega) = |\{k \in \mathbb{N} \mid x_k \in \Omega\}|.$$

(iii) Show that $(\mathcal{H}, \mathscr{F}, \rho)$ is a causal fermion system of spin dimension one.

Exercise 6.2: On the trace constraint

Let $(\mathcal{H}, \mathscr{F}, \rho)$ be a causal fermion system of spin dimension n.

(i) Let $x \in \mathscr{F}$. Show that $\mathcal{L}(x,x) > 0$ whenever $\mathrm{tr}(x) \neq 0$.

(ii) Assume that $\int_{\mathscr{F}} \mathrm{tr}(x)\, d\rho(x) \neq 0$. Show that $\mathcal{S}(\rho) > 0$.

Exercise 6.3: Well-posedness of the causal action principle

This exercise explains why the causal action principle is ill-posed if $\dim \mathscr{H} = \infty$ and $\rho(\mathscr{F}) < \infty$.

(i) Let $\mathscr{H}_0$ be a finite-dimensional Hilbert space of dimension n and $(\mathscr{H}_0, \mathscr{F}_0, \rho_0)$ be a causal fermion system of finite total volume $\rho_0(\mathscr{F}_0) < \infty$. Let $\imath : \mathscr{H}_0 \to \mathscr{H}$ be an isometric embedding of Hilbert spaces. Construct a causal fermion system $(\mathscr{H}, \mathscr{F}, \rho)$ which has the same action, the same total volume and the same values for the trace and boundedness constraints as the causal fermion system $(\mathscr{H}_0, \mathscr{F}_0, \rho_0)$.

(ii) Let $\mathscr{H}_1 := \mathscr{H}_0 \oplus \mathscr{H}_0$. Construct a causal fermion system $(\mathscr{H}_1, \mathscr{F}_1, \rho_1)$ which has the same total volume and the same value of the trace constraint as $(\mathscr{H}_0, \mathscr{F}_0, \rho_0)$ but a smaller action and a smaller value of the boundedness constraint.

Hint: Let $F_{1/2} : L(\mathscr{H}_0) \to L(\mathscr{H}_1)$ be the linear mappings

$$F_1(A)(u \oplus v) := (Au) \oplus 0, \quad F_2(A)(u \oplus v) := 0 \oplus (Au).$$

Show that $F_{1/2}$ maps $\mathscr{F}_0$ to $\mathscr{F}_1$. Define the measure ρ_1 by

$$\rho_1 = \frac{1}{2}\big((F_1)_* \rho_0 + (F_2)_* \rho_0\big).$$

(iii) Iterate the construction in (ii) and apply (i) to obtain a series of universal measures on $\mathscr{F}$ of fixed total volume and with fixed value of the trace constraint, for which the action and the values of the boundedness constraint tend to zero. Do these universal measures converge? If yes, what is the limit?

Exercise 6.4: A causal fermion system on ℓ_2 - part 2.

Referring to Exercise 2.3, show that the support of ρ consists precisely of all the operators x_k. What is spacetime M? What is the causal structure on M? What is the resulting causal action?

Exercise 6.5: On the spectrum of the closed chain

This exercise is devoted to analyzing general properties of the spectrum of the closed chain.

(i) Let x, y be symmetric operators of finite rank on a Hilbert space $\mathscr{H}$. Show that there is a finite-dimensional subspace $\mathscr{K} \subset \mathscr{H}$ on which both x, y are invariant. By choosing an orthonormal basis of $\mathscr{K}$ and restricting the operators on $\mathscr{K}$, we may represent x and y by Hermitian matrices. Therefore, the remainder of this exercise is formulated for simplicity in terms of Hermitian matrices.

(ii) Show that for any matrix Z, the characteristic polynomials of Z and of its adjoint $Z^\dagger$ (being the transposed complex conjugated matrix) are related by complex conjugation, i.e.

$$\det(Z^\dagger - \bar{\lambda}\,\mathbb{I}) = \overline{\det(Z - \lambda\,\mathbb{I})}.$$

(iii) Let X and Y be symmetric matrices. Show that the characteristic polynomials of the matrices XY and YX coincide.

(iv) Combine (*ii*) and (*iii*) to conclude that the characterisstic polynomial of XY has real coefficients, i.e. $\det(XY - \bar{\lambda}\,\mathbb{I}) = \overline{\det(XY - \lambda\,\mathbb{I})}$. Infer that the spectrum of the matrix product XY is symmetric about the real axis, i.e.

$$\det(XY - \lambda\,\mathbb{I}) = 0 \quad \Longrightarrow \quad \det(XY - \bar{\lambda}\,\mathbb{I}) = 0.$$

Exercise 6.6: Regular points

Let $x \in \mathscr{F}$ have $p(x) \leq n$ negative and $q(x) \leq n$ positive eigenvalues. The couple $(p(x), q(x))$ is called the *signature* of x. The operator x is said to be *regular* if $\mathrm{sign}(x) = (n, n)$. The goal of this exercise is to show that the set $\mathscr{F}^{\mathrm{reg}}$ of regular points is open in $\mathscr{F}$. Let us define the *positive* and *negative components* of x as the operators

$$x_\pm := \frac{x \pm |x|}{2}, \quad |x| := \sqrt{x^2}.$$

From the functional calculus it follows that $x\,|x| = |x|\,x$. Prove the following statements.

(i) *Bonus:* Let $\{e_i\ ,\ i = 1, \ldots, m\}$ be an orthogonal set. Any vector set $\{h_i\ ,\ i = 1, \ldots, m\}$ which fulfills the following condition is linearly independent,

$$\|e_i - h_i\| < \frac{\inf\{\|e_i\|\ ,\ i = 1, \ldots, m\}}{m} \quad \text{for all } i = 1, \ldots, m.$$

(ii) For every $x \in \mathscr{F}$,

$$x(\mathrm{im}\, x_\pm) \subset \mathrm{im}\, x_\pm \quad \text{and} \quad x_+\, x_- = 0.$$

Moreover, $x|_{\mathrm{im}\, x_-}$ and $x|_{\mathrm{im}\, x_+}$ are negative and positive definite, respectively.

(iii) Let $x_0 \in \mathscr{F}$. Then there is an orthonormal set $\{e_i \mid i = 1 \ldots \dim S_{x_0}\}$ of eigenvectors of x_0 such that

$$\langle e_i|\, x_0\, e_i\rangle < 0 \text{ for all } i \leq p(x_0), \quad \langle e_i|\, x_0\, e_i\rangle > 0 \text{ for all } p(x_0) < i \leq q(x_0).$$

(iv) The following functions are continuous,

$$f_i : B_r(x_0) \ni x \mapsto f_i(x) := \begin{cases} x_-\, e_i & i \leq p(x_0) \\ x_+\, e_i & p(x_0) < i \leq p(x_0) + q(x_0) \end{cases}.$$

Hint: You can use the general inequaliy $\||A| - |B|\| \leq \|A^2 - B^2\|$

(v) There is a $r > 0$ such that $p(x) \geq p(x_0)$ and $q(x) \geq q(x_0)$ for every $x \in B_r(x_0)$.

Hint: Use the statements above.

(vi) Conclude that $\mathscr{F}^{\mathrm{reg}}$ is an open subset of $\mathscr{F}$.