

Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 5

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

- (i) What is a manifold? What is a chart and an atlas?
- (ii) Illustrate the structure of a vector bundle by a commutative diagram.

Exercises

Exercise 5.1: (A simple manifold)

Let $\mathcal{M} = S^1 \subset \mathbb{C}$ be the unit circle (considered as a subset of the complex plane). We choose the charts (ϕ_1, U_1) and (ϕ_2, U_2) with

$$\begin{aligned} U_1 &= \left\{ e^{i\alpha} \mid \alpha \in \left(-\frac{3\pi}{2}, \frac{3\pi}{2} \right) \right\}, & \phi_1(e^{i\alpha}) &= \alpha \\ U_2 &= \left\{ -e^{i\alpha} \mid \alpha \in \left(-\frac{3\pi}{2}, \frac{3\pi}{2} \right) \right\}, & \phi_2(-e^{i\alpha}) &= \alpha. \end{aligned}$$

Show that these two charts define a smooth atlas of \mathcal{M} .

Exercise 5.2: (A simple vector bundle)

Let $\mathcal{B} = \mathbb{R} \times S^1$ be the two-dimensional cylinder, $\mathcal{M} = S^1$ and

$$\pi : \mathcal{B} \rightarrow \mathcal{M}, \quad (t, e^{i\alpha}) \mapsto e^{i\alpha}.$$

Show that \mathcal{B} is a smooth vector bundle with fiber $Y = \mathbb{R}$.

Exercise 5.3: (Another vector bundle)

Use the mapping

$$F : \mathbb{R} \times S^2 \rightarrow \mathbb{R}^3 \setminus \{0\}, \quad (t, x) \mapsto e^t x$$

(where we consider S^2 as the unit sphere embedded in \mathbb{R}^3) in order to give $\mathcal{B} = \mathbb{R}^3 \setminus \{0\}$ the structure of a vector bundle on S^2 with fiber $Y = \mathbb{R}$.