Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 4

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) How does a scalar product give rise to a norm, metric and topology?

(ii) What is a bounded operator?

(iii) How can one diagonalize a symmetric operator of finite rank?

(iii) What is the topology on the Schwartz space? What are tempered distributions?

(iv) What is the Fourier transform of a Schwartz function and a tempered distribution?

Exercises

Exercise 4.1: (Norm of a scalar product space)
Given a scalar product space $(V, \langle .|. \rangle)$, show that $\|u\| := \sqrt{\langle u|u \rangle}$ defines a norm.

Exercise 4.2: (Norm of a scalar product space)
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Exercise 4.3: (Completeness of $\mathrm{L}(V, W)$)

(a) Show that the operator norm on $\mathrm{L}(V, W)$ is indeed a norm.

(b) Show that $\mathrm{L}(V, W)$ is complete if and only if W is complete.

Exercise 4.4: (Orthogonal complement of a finite-dimensional subspace)

(a) Let I be a finite-dimensional subspace of the Hilbert space $(\mathscr{H}, \langle .|. \rangle)$. Show that its orthogonal complement $I^\perp$ is again a complex vector space.

(b) Show that restricting the scalar product to I, one gets again a Hilbert space. In particular, why is it again complete?

(c) Show that every vector $u \in \mathscr{H}$ has a unique decomposition of the form

$$u = u^{||} + u^\perp \qquad \text{with} \qquad u^{||} \in I, u^\perp \in I^\perp .$$

Hint: Choosing an orthonormal basis $e_1, \ldots, e_n$ of I, a good ansatz for $u^{||}$ is

$$u^{||} = \sum_{k=1}^{n} \langle e_k | u \rangle \, e_k .$$

Exercise 4.5: (Orthogonal projection to closed subspaces of a Hilbert space)
Let $(\mathscr{H}, \langle .|. \rangle)$ be a Hilbert space and $V \subset \mathscr{H}$ a closed subspace.

(a) Show that parallelogram identity: For all $u, v \in \mathscr{H}$,

$$\|u+v\|^2 + \|u-v\|^2 = 2\,\|u\|^2 + 2\,\|v\|^2 \,.$$

(b) Given $u \in \mathscr{H}$, let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V which is a minimizing sequence of the distance to u, i.e.

$$\|u - v_n\| \to \inf_{v\in V} \|u - v\| \,.$$

Prove that the sequence $(v_n)_{n\in\mathbb{N}}$ converges. *Hint:* Apply the parallelogram identity to show that the sequence is Cauchy. Then make use of the completeness of the Hilbert space.

(c) Show that the limit vector $v := \lim_{n\to\infty} v_n$ has the property

$$\langle u - v, w \rangle = 0 \qquad \text{for all } w \in V \,.$$

In view of this equation, the vector v is also referred to as the orthogonal projection of u to V.

Exercise 4.6: (Proof of the Fréchet-Riesz theorem)
Let $\phi \in \mathscr{H}^*$ be non-zero.

(a) Show that the kernel of ϕ is a closed subspace of $\mathscr{H}$.

(b) Apply the result of the previous exercise to construct a nonzero vector v which is orthogonal to $\ker \phi$. Show that this vector is unique up to scaling.

(c) Show that, after a suitable scaling, the vector v satisfies the identity

$$\phi(u) = \langle v|u \rangle \qquad \text{for all } u \in \mathscr{H} \,.$$

(d) Show that the last identity determines the vector v uniquely.

Exercise 4.7: (Multiplication operators)
Let $f \in C^0(\mathbb{R}, \mathbb{C})$ be a continuous, complex-valued function. Assume that it is bounded, i.e. that $\sup_{\mathbb{R}} |f| < \infty$. We consider the multiplication operator M_f on the Hilbert space $\mathscr{H} = L^2(\mathbb{R})$, i.e.

$$M_f \;:\; \mathscr{H} \to \mathscr{H} \,, \qquad (M_f\,\phi)(x) = f(x)\,\phi(x) \,.$$

(a) Show that M_f is a bounded operator, and that its operator norm is given by

$$\|M_f\| = \sup_{\mathbb{R}} |f| \,.$$

(b) Show that M_f is symmetric if and only if f is real-valued. Under which assumptions on f is M_f unitary?

Exercise 4.8: (Dirac sequence)
Given $\varepsilon > 0$, consider the Gaussian

$$\eta_\varepsilon(x) := \frac{1}{\sqrt{4\pi\varepsilon}}\, e^{-\frac{x^2}{4\varepsilon}} \,.$$

(a) Show that ε is a Schwartz function.

(b) Show that the corresponding regular distribution converges to the δ distribution in the sense that for all $f \in \mathcal{S}(\mathbb{R})$,

$$\lim_{\varepsilon \searrow 0} T_{\eta_\varepsilon}(f) = \delta(f)$$

(the δ-distribution is defined by $\delta(f) = f(0)$ for all $f \in \mathcal{S}(\mathbb{R})$).

Exercise 4.9: (Fourier inversion formula)

Prove the Fourier inversion formula for tempered distributions

$$\mathcal{F} \circ \mathcal{F}^* = \mathcal{F}^* \circ \mathcal{F} = \mathbb{1}_{\mathcal{S}'(\mathbb{R}^n)} \; : \; \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \, .$$

Hint: Use the Fourier inversion formula for Schwartz functions together with the definition of the Fourier transform of a tempered distribution.