Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 3

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the topology? What are open and closed sets? What is a compact set?

(ii) Make yourself familiar with the topological definition of continuity. How is it related with the usual notion of continuity using the ε/δ-definition?

(iii) What is a σ-algebra? What is a Borel set? Verify that the Borel sets form a σ-algebra.

Exercises

Exercise 3.1: (Closed sets)

Show that the closed sets (defined as the complements of the open sets) have the following properties:

(i) The sets $\varnothing$ and E are closed.

(ii) Closedness under finite unions:

$$A_1, \ldots, A_n \text{ closed} \quad \Longrightarrow \quad A_1 \cup \cdots \cup A_n \text{ closed} .$$

(iii) Closedness under arbitrary intersections:

$$A_\lambda \text{ closed} \quad \forall \lambda \in \Lambda \quad \Longrightarrow \quad \bigcap_{\lambda \in \Lambda} A_\lambda \text{ closed} .$$

Exercise 3.2: (Completion of a metric space)

This exercise is concerned with the abstract completion of a metric space. Consider the set of Cauchy sequences in E endowed with the distance function

$$d\Big(\big(x_n\big)_{n \in \mathbb{N}}, \big(y_n\big)_{n \in \mathbb{N}} \Big) := \lim_{n \to \infty} d(x_n, y_n) . \tag{1}$$

(i) Show that the limit in (1) exists. Show that the so-defined distance function has all the properties of a metric.

(ii) Show that

$$\big(x_n\big)_{n \in \mathbb{N}} \simeq \big(y_n\big)_{n \in \mathbb{N}} \qquad \text{if} \qquad d\Big(\big(x_n\big)_{n \in \mathbb{N}}, \big(y_n\big)_{n \in \mathbb{N}} \Big) = 0$$

is an equivalence relation. Why is the distance function well-defined on the equivalence classes?

Exercise 3.3: (Borel algebra)

This exercise is devoted to the constrution of the Borel algebra.

(a) Show that the power set of $\mathscr{F}$ (i.e. the set of all subsets) forms a σ-algebra.

(b) Show that the intersection of σ-algebras is again a σ-algebra.

(c) Combine (a) and (b) to conclude that there is a smallest σ-algebra which contains all open subsets of $\mathscr{F}$.

Exercise 3.4: (Understanding the push-forward measure)

The purpose of this aufgabe is to introduce the so-called push-forward measure, which will be used later for the construction of causal fermion systems. Let $\mathscr{M} \subset \mathbb{R}^3$ be a smooth surface described by a parametrization Φ. More precisely, given an open subset $\Omega \subset \mathbb{R}^2$, we consider a smooth injective map

$$\Phi \,:\, \Omega \to \mathbb{R}^3$$

with the property that $D\Phi|_p : \mathbb{R}^2 \to \mathbb{R}^3$ has rank two for all $p \in \Omega$. Then the surface $\mathscr{M}$ is defined as the image $\Phi(\Omega) \subset \mathbb{R}^3$. We now introduce a measure ρ on $\mathbb{R}^3$ as the *push-forward measure* of the Lebesgue measure on $\mathbb{R}^2$ through Φ: Let μ be the Lebesgue measure on $\mathbb{R}^2$. We define a set $U \subset \mathbb{R}^3$ to be ρ-measurable if and only if its pre-image $\Phi^{-1}(U) \subset \mathbb{R}^2$ is μ-measurable. On the ρ-measurable sets we define the measure ρ by

$$\rho(U) = \mu\big(\Phi^{-1}(U)\big)\,.$$

Verify that the ρ-measurable sets form a σ-algebra, and that ρ is a measure. What are the sets of ρ-measure zero? What is the support of the measure ρ?

Suppose that Φ is no longer assumed to be injective. Is ρ still a well-defined measure? Is ρ well-defined if Φ is only assumed to be continuous? What are the minimal regularity assumptions on Φ needed for the push-forward measure to be well-defined? What is the support of ρ in this general setting?