Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 22

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the partition function and the quantum state of a causal fermion system?

(ii) How are these notions related to those of perturbative quantum field theory?

Exercises

Exercise 22.1:

The purpose of this exercise is to get familiar with the notion of a *quantum state* defined as a positive linear mapping

$$\omega : \mathcal{A} \to \mathbb{C} \qquad \text{with} \qquad \omega(A^* A) \geq 0 \quad \text{for all } A \in \mathcal{A}$$

(here for the algebra $\mathcal{A}$ we take the *algebra of observables*, i.e. the set of all operators obtained from all observables by taking products and linear combinations). In quantum mechanics, the system is usually described by a unit vector ψ in a Hilbert space $(\mathscr{H}, \langle .|. \rangle)$. An observable corresponds to a symmetric operator $A \in \mathrm{L}(\mathscr{H})$ on this Hilbert space (for simplicity, we here restrict attention to bounded operators). The expectation value of a measurement is given by the expectation $\langle \psi | A | \psi \rangle$.

(i) Show that the linear operator $W \in \mathrm{L}(\mathscr{H})$ defined by

$$W\phi = \langle \phi | \psi \rangle \, \psi \qquad \text{or, in bra/ket notation,} \qquad W = |\psi\rangle\langle\psi| \tag{1}$$

is a projection operator (i.e. it is symmetric and idempotent). Show that the expectation value of a measurement can be written as

$$\langle \psi | A | \psi \rangle = \mathrm{tr}_{\mathscr{H}} \big(WA \big) \,.$$

(ii) Show that the mapping

$$\omega \;:\; A \mapsto \mathrm{tr}_{\mathscr{H}} \big(WA \big) \tag{2}$$

is a quantum state.

(iii) Let ψ_1 and ψ_2 be two distinct, non-zero vectors of $\mathscr{H}$. Show that, choosing

$$W := |\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| \,, \tag{3}$$

the mapping (2) again defines a quantum state in the sense (**??**). Show that this quantum state cannot be written in the form (1). One refers to (1) as a *pure state*, whereas (3) is a *mixed state*.

(iv) Is the quantum state in (iii) properly normalized in the sense that $\omega(\mathbb{1}) = 1$? If not, how can this normalization be arranged?