Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 21

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the connection between the linearized field equations and the dynamical wave equation?

(ii) In which sense does the dynamical wave equation correspond to and generalize the Dirac equation? What corresponds to the Dirac current conservation?

Exercises

Exercise 21.1: Representation of the commutator inner product

The goal of this exercise is to show that the commutator inner product

$$\mathscr{C}_\rho^\Omega(u) := \int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y)\, \big(D_{1,\mathrm{J}(u)} - D_{2,\mathrm{J}(u)}\big)\mathcal{L}(x,y) \qquad \text{with } u \in \mathscr{H} \tag{1}$$

gives rise to a sesquilinear form on the Hilbert space of the form

$$\langle u|v\rangle_\rho^\Omega := -2i \left(\int_\Omega d\rho(x) \int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x) \int_\Omega d\rho(y) \right) \prec\!\psi^u(x)\,|\, Q(x,y)\,\phi^v(y)\!\succ_x . \tag{2}$$

(i) Show that first variations of the Lagrangian can be written as

$$\delta\mathcal{L}(x,y) = 2\,\mathrm{Re}\,\mathrm{Tr}_{S_xM}\big(Q(x,y)\,\delta P(y,x)\big)$$

with a suitable kernel $Q(x,y) : S_y \to S_x$. Show that this kernel can be chosen to be symmetric, i.e. that $Q(x,y)^* = Q(y,x)$.

(ii) Show that the variation described by the commutator jet

$$v(x) = i\big[\mathscr{A}, x\big] \qquad \text{and} \qquad \mathscr{A}u := \langle u|\psi\rangle_{\mathscr{H}}\,\psi$$

corresponds to the variation of the integrand in (1)

$$\big(D_{1,\mathrm{J}(u)} - D_{2,\mathrm{J}(u)}\big)\mathcal{L}(x,y) = -2i\big(i \prec\!\psi(x)\,|\,Q(x,y)\,\psi(y)\!\succ_x - i \prec\!\psi(y)\,|\,Q(y,x)\,\psi(x)\!\succ_y\big) .$$

(iii) Use the polarization formula

$$\langle u|v\rangle_\rho^\Omega := \frac{1}{4}\Big(\mathscr{C}_\rho^\Omega(u+v) - \mathscr{C}_\rho^\Omega(u-v)\Big) - \frac{i}{4}\Big(\mathscr{C}_\rho^\Omega(u+iv) - \mathscr{C}_\rho^\Omega(u-iv)\Big)$$

to conclude that $\langle u|v\rangle_\rho^\Omega$ has the desired representation (2).

Exercise 21.2: Extending the commutator inner product

The goal of this exercise is to illustrate how the commutator inner product can be extended to more general wave functions. To this end, assume that we are given a space of wave function $\mathscr{W}$ which all satisfy the dynamical wave equation

$$\int_M Q^{\text{dyn}}(x,y)\,\psi(y)\,d\rho(y) = 0$$

with a suitable kernel $Q^{\text{dyn}}(x,y)$. Prove that, under these assumptions, the inner product

$$\langle\psi|\phi\rangle_\rho^\Omega := -2i\left(\int_\Omega d\rho(x)\int_{M\setminus\Omega} d\rho(y) - \int_{M\setminus\Omega} d\rho(x)\int_\Omega d\rho(y)\right) \prec\!\psi(x)\,|\,Q^{\text{dyn}}(x,y)\,\phi(y)\!\succ_x$$

is conserved for all $\psi, \phi \in \mathscr{W}$. *Hint:* In a first step it seem a good idea to choose $\Omega = \Omega_t$ as the past of an equal time hypersurface and to differentiate with respect to t. More generally, one can consider the difference of (2) for two sets Ω and Ω' which differ by a compact set.