Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 20

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) Why is the construction spin connection related to a polar decomposition? Of which operator?

(ii) In which way does the construction go beyond a polar decomposition? Why is this needed?

(iii) What is the curvature of the spin connection?

(iv) How are these notions of connection and curvature related to the classical notions of spin geometry?

Exercises

Exercise 20.1: Signature of Clifford extensions

(i) Let $\mathcal{T}^{s_x}$ be a Clifford extension of the Euclidean sign operator s_x. Show that resulting bilinear form $\langle .,. \rangle$ on $\mathcal{T}^{s_x}$ is Lorentzian, i.e. that it has signature $(1, k)$ with $k \in \mathbb{N}$. *Hint:* it is most convenient to work in an orthonormal eigenvector basis of the Euclidean sign operator. You also find the proof in [**?**, Lemma 4.4].

(ii) Now let $\mathcal{T}^v$ be a the Clifford extension of a general sign operator v. Is the signature of $\langle .,. \rangle$ necessarily Lorentzian? *Hint:* It may be helpful to have a look at [**?**, Lemma 3.2].

Exercise 20.2: Clifford extensions on the Dirac sphere

We return to the Dirac sphere considered in Exercise **??**. Thus we let $F : S^2 \to \mathscr{F}$ and $M := \text{supp}\rho = F(S^2)$.

(i) Given $p \in S^2$, we consider the spacetime point $x = F(p) \in M$. Construct the Euclidean sign operator s_x at x.

(ii) What is the maximal dimension of Clifford extensions of the Euclidean sign operator? Show that the Clifford extension of maximal dimension is unique.

(iii) Give an explicit parametrization of this Clifford extension. How does the inner product $\langle .,. \rangle$ look like in your parametrization?

Exercise 20.3: Stability of the causal structure

A binary relation P on $\mathscr{F}$ is said to be *stable under perturbations* if

$$(x_0, y_0) \in P \implies \exists\, r > 0 \;:\; B_r(x_0) \times B_r(y_0) \subset P.$$

Following Definition **??**, two points $x, y \in \mathscr{F}$ are said to be *properly timelike* separated if the closed chain A_{xy} has a strictly positive spectrum and if all eigenspaces are definite subspaces of $(S_x, \prec\cdot\,,\,\cdot\succ)$.

(i) Show that proper timelike separation implies timelike separation.

(ii) Show by a counterexample with 3×3 matrices that the notion of timelike separation is *not* stable under perturbations.

(iii) Show that the notion of properly timelike separation is stable under perturbations.