Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 2

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) How is the Hilbert space of Dirac solutions at a given time t constructed? How does unitary time evolution follow from the conservation of the Dirac current? What are the differences and the similarities between the Hilbert space and the Krein space?

(ii) What are the upper and lower mass shells? How are these connected to the concept of the Dirac sea?

(iii) What is pair creation? What is the electric charge and energy of an anti-particle (i.e. a "hole" in the sea).

(iv) What are the problems of the naive Dirac sea picture?

Exercises

Exercise 2.1: On left- and right-handed spinors

Out of the Dirac matrices γ^i one can form the *pseudoscalar matrix* γ^5 by

$$\gamma^5 := \frac{i}{4!}\epsilon_{jklm}\,\gamma^j\,\gamma^k\,\gamma^l\,\gamma^m = i\,\gamma^0\gamma^1\gamma^2\gamma^3,$$

where ϵ_{jklm} is the totally antisymmetric symbol. Using the commutation relations of the Dirac matrices, shows that $(\gamma^5)^\dagger = \gamma^5 = -(\gamma^5)^*$, $(\gamma^5)^2 = \mathbb{I}_4$ and $\{\gamma^5, \gamma^i\} := \gamma^5\gamma^i + \gamma^i\gamma^5 = 0$. As a corollary, show that the matrices

$$\chi_L := \frac{1}{2}(\mathbb{I}_4 - \gamma^5), \quad \chi_R := \frac{1}{2}(\mathbb{I}_4 + \gamma^5),$$

called the *chiral projectors*, satisfy the following identities

$$\begin{gathered}(\chi_{L/R})^2 = \chi_{L/R}, \quad \gamma^5\,\chi_L = -\chi_L, \quad \gamma^5\,\chi_R = \chi_R, \quad (\chi_L)^* = \chi_R, \\ (\chi_{L/R})^\dagger = \chi_{L/R}, \quad \chi_L + \chi_R = \mathbb{I}_4.\end{gathered} \tag{1}$$

Given a *spinor* $\psi \in \mathbb{C}^4$, the projections $\chi_L\,\psi$ and $\chi_R\,\psi$ are referred to as the *left-* and *right-handed components* of the spinor, respectively. Show that

$$\gamma^j\,\chi_{L/R} = \chi_{R/L}\,\gamma^j \quad \text{for all } j \in \{0,1,2,3\}.$$

Using these relations, show that the Dirac equation with an external electromagnetic field A_j can be rewritten as a system of equations for the left- and right-handed components of ψ,

$$i\gamma^k\,(\partial_k - iA_k)\,\chi_L\psi = m\,\chi_R\psi, \quad i\gamma^k\,(\partial_k - iA_k)\,\chi_R\psi = m\,\chi_L\psi$$

What happens in the limiting case $m = 0$?

Exercise 2.2: Invariance of the inner product under Lorentz transformations

Let $(\mathbb{R}^4, \eta)$ be Minkowski Space, where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. The matrix Lie group

$$\mathcal{L}_+^\uparrow := \left\{\ \Lambda \in \mathrm{Mat}(4, \mathbb{R})\ \mid\ \eta_{ij}\,\Lambda^i{}_l\,\Lambda^j{}_m = \eta_{lm},\ \Lambda^0{}_0 > 0,\ \det\Lambda = +1\ \right\}.$$

is known as the *proper orthochronous Lorentz group*. A *Cauchy hyperplane* is a 3-dimensional hyperplane $\mathcal{N} \subset \mathbb{R}^4$ with the property that its tangent vectors are space-like. A vector field $\nu \in T\mathbb{R}^4$ is said to be *normal* to $\mathcal{N}$ and *future directed* if, for all $p \in \mathcal{N}$,

$$\eta_{ij}\nu^i\nu^j|_p = 1, \quad \nu(p)^0 > 0 \quad \text{and} \quad \eta_{ij}\,\nu^i\,u^j|_p = 0 \ \text{ for all } u \in T\mathcal{N}.$$

On the manifold $\mathcal{N}$ we define a *volume form*

$$d\,\mathrm{vol}_\mathcal{N} := \iota_\nu(d\,\mathrm{vol}_{\mathbb{R}^4}) = \nu^{i_0}\,\epsilon_{i_0 i_1 i_2 i_3}\,dx^{i_1}\otimes dx^{i_2}\otimes dx^{i_3},$$

where ϵ_{ijkl} is the totally antisymmetric symbol and $d\,\mathrm{vol}_{\mathbb{R}^4} = \epsilon_{i_0 i_1 i_2 i_3}\,dx^{i_0}\otimes dx^{i_1}\otimes dx^{i_2}\otimes dx^{i_3}$ is the canonical volume form on $\mathbb{R}^4$.

Finally, consider the set of smooth spatially compact solutions of the Dirac equation

$$\mathcal{D} := \left\{\psi \in C^\infty_{\mathrm{sc}}(\mathbb{R}^4, \mathbb{C}^4)\mid (i\gamma^j\,\partial_j - m)\psi(x) = 0\right\}.$$

Prove the following statements.

(i) The normal vector field ν is time-like, i.e. $\eta_{ij}\,\nu^i\,\nu^j > 0$.

(ii) *Current conservation*: for all $\psi \in \mathcal{D}$,

$$\partial_k J^k = 0, \quad \text{with} \quad J^k(x) := \prec\!\psi(x), \gamma^k\,\psi(x)\!\succ.$$

(iii) The following integral is independent of $\mathcal{N}$, with $\nu_i := \eta_{ij}\,\nu^j$,

$$\langle\psi, \phi\rangle := \int_\mathcal{N} \prec\!\psi(x), \gamma^i\,\phi(x)\!\succ \nu_i(x)\,d\mathrm{vol}_\mathcal{N}(x), \quad \psi, \phi \in \mathcal{D}. \tag{2}$$

Hint: Gauß Theorem?

(iv) The volume form $d\,\mathrm{vol}_\mathcal{N}$ is invariant under Lorentz transformations, i.e.

$$f_\Lambda^*(d\,\mathrm{vol}_{\hat{\mathcal{N}}}) = d\,\mathrm{vol}_\mathcal{N}, \quad \hat{\mathcal{N}} = f_\Lambda(\mathcal{N}),$$

where $f_\Lambda : \mathbb{R}^4 \ni x \mapsto \Lambda\,x \in \mathbb{R}^4$ and ϕ^* denotes the pull-back.

Hint: What is the relation between ϵ_{jklm} and the determinant of a 4×4 Matrix?

(v) The following linear mapping is well-defined,

$$\mathcal{D} \ni \psi \mapsto \hat{\psi} := U(\Lambda^{-1})\psi(\Lambda\,\cdot\,) \in \mathcal{D}, \tag{3}$$

where $U : \mathcal{L}_+^\uparrow \ni \Lambda \mapsto U(\Lambda) \in \mathrm{Mat}(4, \mathbb{C})$ satisfies

$$U(\Lambda^{-1}) = U(\Lambda)^* = U(\Lambda)^{-1}, \quad U(\Lambda)^*\,\gamma^i\,U(\Lambda) = \Lambda^i{}_j\,\gamma^j.$$

Moreover, the product (2) is invariant under (3).