Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 19

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is a topological fermion system? What is a Riemannian fermion system?

(ii) How are the structures of a spinor bundle be generalized in the notion of a topological spinor bundle?

(iii) How do the notions of "spin space," "Clifford subspace" and "Clifford multiplication" relate to the more general structures of a topological fermion system?

Exercises

Exercise 19.1: Vector fields on a closed Riemannian manifold

Let $(\mathscr{M}, g)$ be a smooth compact Riemannian manifold of dimension k and Δ the covariant Laplacian on smooth vector fields. We complexify the vector fields and endow them with the L^2-scalar product

$$\langle u|v\rangle_{L^2} := \int_{\mathscr{M}} g_{jk}\, \overline{u^j}\, v^k \, d\mu_{\mathscr{M}} \;, \tag{1}$$

where $d\mu_{\mathscr{M}} = \sqrt{\det g}\, d^kx$ is the volume measure on $\mathscr{M}$. Show the following:

(i) The operator $-\Delta$ is essentially self-adjoint and has smooth eigenfunctions.

(ii) We choose a parameter $L > 0$ and choose $\mathscr{H}$ as the spectral subspace of the Laplacian

$$\mathscr{H} = \operatorname{rg} \chi_{[0,L]}(-\Delta) \;.$$

Show that $\mathscr{H}$ is finite-dimensional.

(iii) For any $p \in \mathscr{M}$ we define the local correlation operator $F(p) \in \mathrm{L}(\mathscr{H})$ by

$$-g_{ij}\, \overline{u^i(p)}\, v^j(p) = \langle u|F(p)v\rangle_{L^2} \qquad \text{for all } u, v \in \mathscr{H} \;.$$

Show that this operator is well-defined, negative semi-definite and has rank at most k.

(iv) We again introduce the measure by $\rho = F_*\mu$. Show that $(\mathscr{H}, \mathscr{F}, \rho)$ is a Riemannian fermion system of spin dimension k.

Exercise 19.2: Spinors on a closed Riemannian manifold

Let $(\mathscr{M}, g)$ be a compact Riemannian spin manifold of dimension $k \geq 1$. Then the spinor bundle $S\mathscr{M}$ is a vector bundle with fibre $S_p\mathscr{M} \simeq \mathbb{C}^n$ with $n = 2^{[k/2]}$ (see for example [**?**, **?**]). Moreover, the spin inner product $\prec .|. \succ_p : S_p\mathscr{M} \times S_p\mathscr{M} \to \mathbb{C}$ is positive definite. On the smooth sections $\Gamma(S\mathscr{M})$ of the spinor bundle we can thus introduce the scalar product

$$\langle \psi | \phi \rangle = \int_{\mathscr{M}} \prec \psi | \phi \succ_p \, d\mu_{\mathscr{M}}(p) \,. \tag{2}$$

Forming the completion gives the Hilbert space $L^2(\mathscr{M}, S\mathscr{M})$.

(i) The Dirac operator $\mathcal{D}$ with domain of definition $\Gamma(S\mathscr{M})$ is an essentially self-adjoint operator on $L^2(\mathscr{M}, S\mathscr{M})$. It has a purely discrete spectrum and finite-dimensional eigenspaces.

(ii) Given a parameter $L > 0$, we let $\mathscr{H}$ be the space spanned by all eigenvectors whose eigenvalues lie in the interval $[-L, 0]$,

$$\mathscr{H} = \text{rg}\, \chi_{[-L,0]}(\mathcal{D}) \subset L^2(\mathscr{M}, S\mathscr{M}) \,.$$

Denoting the restriction of the L^2-scalar product to $\mathscr{H}$ by $\langle .|. \rangle_{\mathscr{H}}$, we obtain a finite-dimensional Hilbert space $(\mathscr{H}, \langle .|. \rangle_{\mathscr{H}})$. Show that this Hilbert space is finite-dimensional and consists of smooth wave functions.

(iii) For every $p \in \mathscr{M}$ we introduce the local correlation operator $F(p)$ by

$$-\prec \psi | \phi \succ_p = \langle \psi | F(p) \phi \rangle_{\mathscr{H}} \qquad \text{for all } \psi, \phi \in \mathscr{H} \,.$$

Show that this operator is negative semi-definite and has rank at most n.

(iv) We again introduce the measure by $\rho = F_* \mu$. Show that $(\mathscr{H}, \mathscr{F}, \rho)$ is a Riemannian fermion system of spin dimension n.