Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 18

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the light-cone expansion?

(ii) How can the light-cone expansion be regularized?

(iii) How does the formalism of the continuum limit work? What is weak evaluation of the light cone?

Exercises

Exercise 18.1:

This exercise explains the notion of the *light-cone expansion* in simple examples.

(i) What is the light-cone expansion of a smooth function on $\mathscr{M} \times \mathscr{M}$? In which sense is it trivial? In which sense is it non-unique?

(ii) Show that $A(x,y) = \log\big(|y-x|^2\big)$ is a well-defined distribution on $\mathscr{M} \times \mathscr{M}$. What is the order on the light cone? Write down a light-cone expansion.

(iii) Now consider the distributional derivatives

$$\Big(\frac{\partial}{\partial x^0}\Big)^p A(x,y) \qquad \text{with} \qquad p \in \mathbb{N}$$

and $A(x,y)$ as in part (ii). What is the order on the light cone? Write down a light-cone expansion.

(iv) Consider the function

$$E(x,y) = \sin\big((y-x)^2\big)\, \log\big(|y-x|^2\big)\,.$$

Determine the order on the light cone and give a light-cone expansion.

(v) Consider the function

$$E(x,y) = \begin{cases} e^{-\frac{1}{(y-x)^2}} & \text{if } (y-x)^2 \geq 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Determine the order on the light cone and give a light-cone expansion.

(vi) Show that the expression

$$\lim_{\varepsilon \searrow 0} \frac{\log\left(|y-x|^2\right)}{(y-x)^4 + i\varepsilon}$$

is a well-defined distribution on $\mathscr{M} \times \mathscr{M}$. Derive its light-cone expansion.

Exercise 18.2: Understanding the light cone expansion

This exercise aims to familiarize you with some of the particularities of the light cone expansion.

(i) Let $A(x,y) := (x-y)^{2k_0}$ with $k_0 \in \mathbb{Z}$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $A(x,y)$ and prove that it is one.

(ii) Let $B(x,y) := (x-y)^{2k_0} + (x-y)^{2k_1}$, where $k_0, k_1 \in \mathbb{Z}$ and $k_0 < k_1$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $B(x,y)$ and prove that it is one.

(iii) Let $C(x,y) := (x-y)^{2k_0} f(x,y) + (x-y)^{2k_1} g(x,y)$, where f and g are smooth functions in x and y and $k0, k1$ as above. Construct a light-cone expansion of $C(x,y)$ and prove that it is one.

(iv) Let $D(x,y) := \sin\left((x-y)^2\right)(x-y)^2$. Use your results from *ii.)* and *iii.)* to construct two different light-cone expansions of $D(x,y)$. Why might this non-uniqueness not be a problem for the scope of the lecture?

(v) Finally, consider the function

$$E(x,y) = \sin\left((y-x)^2\right) + \begin{cases} e^{-\frac{1}{(y-x)^2}} & \text{if } (y-x)^2 \geq 0 \\ 0 & \text{else} \end{cases},$$

Determine its order(s) on the light cone and a light cone expansion. (Prove your answer.)

Hint: For (iv) and (v): Expand the sine function.

Exercise 18.3:

This exercise explains in a simple example how the *regularization of the light-cone expansion* works.

(i) Consider the singular term of the first summand of the light-cone expansion in Minkowski space,

$$\lim_{\nu \searrow 0} \frac{1}{\xi^2 - i\nu\, \xi^0} \tag{1}$$

(where again $\xi := y - x$). A simple method to remove the pole is not to take the limit $\nu \searrow 0$, but instead to set $\nu = 2\varepsilon$,

$$\frac{1}{\xi^2 - 2i\varepsilon\, \xi^0}\,. \tag{2}$$

Show that this regularization can be realized by the replacement

$$\xi^0 \to \xi^0 - i\varepsilon\,,$$

up to a multiplicative error of the order

$$\left(1 + \mathscr{O}\Big(\frac{\varepsilon^2}{\xi^2}\Big)\right). \tag{3}$$

The basic concept behind the regularized Hadamard expansion is to regularize all singular terms in this way, leaving all smooth functions unchanged. This gives a consistent formalism is one works throughout with error terms of the form (3). *Hint:* This is the so-called $i\varepsilon$-regularization introduced in the "blue book" in Section 2.4.

(ii) Show that for kernels written as Fourier transforms

$$K(x,y) = \int_M \frac{d^4p}{(2\pi)^4}\, \hat{K}(p)\, e^{-ip(y-x)}$$

(with $\hat{K}$ supported in say the lower half plane $\{p^0 < 0\}$), the replacement rule (2) amounts to inserting a convergence-generating factor $e^{\varepsilon p^0}$ into the integrand.

Exercise 18.4:

The goal of this exercise is to explore *weak evaluation on the light cone* in a simple example.

(i) Show that, setting $t = \xi^0$ and choosing polar coordinates with $r = |\vec{\xi}|$, regularizing the pole in (1) according to (2) gives the function

$$\frac{1}{(t-i\varepsilon)^2 - r^2}\,.$$

(ii) As a simple example of a composite expression, we take the absolute square of the regualarized function

$$\frac{1}{\big|(t-i\varepsilon)^2 - r^2\big|^2}\,. \tag{4}$$

Show that this expression is ill-defined in the limit $\varepsilon \searrow 0$ even as a distribution.

(iii) Use the identity

$$\frac{1}{(t-i\varepsilon)^2 - r^2} = \frac{1}{(t-i\varepsilon-r)(t-i\varepsilon+r)} = \frac{1}{2r}\left(\frac{1}{t-i\varepsilon-r} - \frac{1}{t-i\varepsilon+r}\right)$$

to rewrite the integrand in (4) in the form

$$\sum_{p,q=0}^{1} \frac{\eta_{p,q}(t,r,\varepsilon)}{(t-i\varepsilon-r)^p\,(t+i\varepsilon-r)^q}\,,$$

with functions $\eta_{p,q}(t,r,\varepsilon)$ which in the limit $\varepsilon \searrow 0$ converge to smooth functions. Compute the functions $\eta_{p,q}$.

(iv) We now compute the leading contributions and specify what we mean by "leading." First compute the following integrals with residues:

$$I_0(\varepsilon) := \int_{-\infty}^{\infty} \frac{1}{(t-i\varepsilon-r)\,(t+i\varepsilon-r)}\, dt\,.$$

Show that

$$\int_{-\infty}^{\infty} \frac{\eta_{1,1}(t,r)}{(t-i\varepsilon-r)^2\,(t+i\varepsilon-r)^2}\, dt = I_0(\varepsilon)\,\eta_{2,2}(r,r) + \mathscr{O}(\varepsilon)\,.$$

Explain in which sense this formula is a special case of the weak evaluation formula in the lecture.