Online Course on Causal Fermion Systems

Prof. Dr. Felix Finster, Dr. Marco Oppio

Guiding Questions and Exercises 17

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the causal perturbation expansion?

(ii) How is causality built in the light-cone expansions?

Exercises

Exercise 17.1: Perturbative description of gauge transformations

We consider the perturbation expansion for the Dirac Green's operators

$$\tilde{s}_m^\wedge = \sum_{n=0}^\infty \big(- s_m^\wedge \,\mathscr{B} \big)^n \, s_m^\wedge$$

for a perturbation by a pure gauge potential, i.e.

$$\mathscr{B}(x) = \partial\!\!\!/ \Lambda(x)$$

with a real-valued function Λ.

(a) Show that the Dirac operator with interaction can be written as

$$i\partial\!\!\!/ + (\partial\!\!\!/ \Lambda) - m = e^{i\Lambda(x)} \big(i\partial\!\!\!/ - m \big)\, e^{-i\Lambda(x)} \:.$$

Conclude that the perturbation of the Dirac solutions amounts to multiplication by a phase function, i.e.

$$\tilde{\Psi}(x) = e^{i\Lambda(x)}\, \Psi(x) \:.$$

Explain why these findings are a manifestation of the local gauge freedom of electrodynamics.

(b) Show that the gauge phases also appear in the perturbation expansion in the sense that

$$\tilde{s}_m^\wedge(x,y) = e^{i\Lambda(x)}\, s_m^\wedge(x,y)\, e^{-i\Lambda(y)} \:.$$

Hint: To first oder, one needs to show that

$$-\big(s_m^\wedge \,\mathscr{B}\, s_m^\wedge \big)(x,y) = i \big(\Lambda(x) - \Lambda(y) \big)\, s_m^\wedge(x,y) \:.$$

To this end, it is convenient to write the perturbation operator as a commutator,

$$\mathscr{B} = -i \left[(i\partial\!\!\!/ - m),\, \Lambda \right]$$

and use the defining equation of the Green's operator

$$(i\partial\!\!\!/_x - m)\, s_m(x,y) = \delta^4(x-y) \:.$$

To higher order, one can proceed inductively.

Exercise 17.2:

Verify by formal computation that in the Minkowski vacuum, the fundamental solution k_m and the Green's operator s_m defined by

$$s_m := \frac{1}{2} \left(s_m^\vee + s_m^\wedge \right) \tag{1}$$

satisfy the distributional relations in the mass parameters m and m'

$$\begin{aligned} k_m \, k_{m'} &= \delta(m - m') \, p_m \\ k_m \, s_{m'} &= s_{m'} \, k_m = \frac{\text{PP}}{m - m'} \, k_m \ , \end{aligned}$$

where PP denotes the principal part, and p_m is the distribution

$$p_m(k) = (\slashed{k} + m) \, \delta(k^2 - m^2) \ . \tag{2}$$

Hint: By a "formal computation" we mean that you do not need to evaluate weakly in the mass with test functions.

Exercise 17.3:

Proceed similar as in Exercise to derive a relation for the operator product $s_m^\vee s_{m'}^\vee$. Derive the relation

$$s_m \, s_{m'} = \frac{\text{PP}}{m - m'} \, (s_m - s_{m'}) + \pi^2 \, \delta(m - m') \, p_m \ .$$