Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 16

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) What is the mass oscillation property? Why is it needed for the construction of the fermionic signature operator?

(ii) How can the kernel of the fermionic projector be constructed from the fermionic signature operator?

Exercises

Exercise 16.1: Towards the mass oscillation property - part 1

This exercise illustrates the mass oscillation property. Let $0 < m_L < m_R$ and $\eta \in C_0^\infty((m_L, m_R))$. Show that the function f given by

$$f(t) = \int_{m_L}^{m_R} \eta(m)\, e^{-i\sqrt{1+m^2}\, t} dm$$

has rapid decay. Does this result remain valid if m_L and m_R are chosen to have opposite signs? Justify your finding by a proof or a counter example.

Exercise 16.2: Towards the mass oscillation property - part 2

Let R_T be the "space-time strip"

$$R_T = \{(t, \vec{x}) \in \mathbb{R}^{1,3} \text{ with } 0 < t < T\}\,.$$

Show that for any solutions $\psi, \phi \in C^\infty_{\text{sc}}(\mathbb{R}^4, \mathbb{C}^4) \cap \mathcal{H}_m$ of the Dirac equation, the following inequality holds,

$$\big|{<}\psi|\phi{>}_T\big| \leq T\, \|\psi\|_m\, \|\phi\|_m\,, \quad \text{where} \quad {<}\psi|\phi{>}_T := \int_{R_T} \prec\!\psi(x)|\phi(x)\!\succ d^4x.$$

This estimate illustrates how in space-times of finite lifetime, the space-time inner product is a bounded sequilinear form on $\mathcal{H}_m$.

Exercise 16.3: Towards the mass oscillation property - part 3

Let R_T again be the "space-time strip" of the previous exercises. Moreover, we again let $\mathcal{H} \subset \mathcal{H}_m$ be a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, i.e.

$$\mathcal{H} \subset C^\infty_{\text{sc}}(\mathbb{R}^4, \mathbb{C}^4) \cap \mathcal{H}_m \qquad \text{finite-dimensional}$$

(see Exercise 10). Show that the fermionic signature operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ defined by

$$<\psi|\phi>_T = (\psi|\mathcal{S}\phi)_m \qquad \text{for all } \psi, \phi \in \mathcal{H}$$

can be expressed within the causal fermion system by

$$\mathcal{S} = -\int_{R_T} x \, d\rho(x)$$

(where ρ is again the push-forward of d^4x).

Exercise 16.4: Bonus: External field problem

In physics, the notion of "particle" and "anti-particle" is often introduced as follows: Solutions of the Dirac equation with positive frequency are called "particles" and solutions with negative frequency "anti-particles". In this exercise, we will check in how far this makes sense.
To this end, take a look at the Dirac equation in an external field:

$$(i\partial\!\!\!/ + \mathcal{B} - m)\psi = 0. \tag{1}$$

Assume that $\mathcal{B}$ is time-dependent and has the following form:

$$\mathcal{B}(t, x) = V\, \Theta(t - t_0)\Theta(t_1 - t),$$

where $V \in \mathbb{R}$, Θ denotes the Heaviside step function and $t_0 = 0,\ t_1 = 1$. In oder to construct a solution thereof, for a given momentum $\vec{k}$, we use plane wave solutions of the Dirac equation,

$$\psi(t, \vec{x}) = e^{-i\omega t + i\vec{k}\vec{x}} \chi_{\vec{k}},$$

where $\chi_{\vec{k}}$ is a spinor $\in \mathbb{C}^4$, and patch them together suitably. (The quantity ω is called the "frequency" or "energy", and $\vec{k}$ the"momentum".) To simplify the calculation, we set $\vec{k} = (k_1, 0, 0)^T$. Proceed as follows:

(i) First, take a look at the region $t < t_0$. Reformulate Eq. (1) s.t. there is only the time derivative on the left hand side. (Hint: Multiply by γ^0.)

(ii) Insert the plane wave ansatz with $\vec{k} = (k_1, 0, 0)^T$ into the equation. Your equation now has the form $\omega\psi = H(k_1)\psi$. Show that the eigenvalues of $H(k_1)$ are $\pm\omega_0$ with $\omega_0 := \sqrt{(k_1)^2 + m^2}$.

(iii) Show that one eigenvector belonging to $+\omega_0$ is $\chi_0^+ := (\frac{m+\omega_0}{k1}, 0, 0, 1)^T$ and that one eigenvector belonging to $-\omega_0$ is $\chi_0^- := (\frac{m-\omega_0}{k1}, 0, 0, 1)^T$. (Both eigenvalues have multiplicity 2, but we don't need the other two eigenvectors here.)

(iv) With this, you have constructed plane wave solutions $e^{-i(\pm\omega_0)t + i\vec{k}\vec{x}}\chi_0^\pm$ for $t < t_0$ and also for $t > t_1$. By transforming $m \to (m - V)$, you immediately obtain plane wave solutions also for $t_0 < t < t_1$. Denote the respective quantities by ω_1 and $\chi_1^\pm$.

(v) Assume that for $t < t_0$ there is one "particle" present, i.e. set

$$\psi(t, \vec{x}) = e^{-i\omega_0 t + i\vec{k}\vec{x}}\chi_0^+ \quad \text{for } t < t_0.$$

Assume that the solution for $t_0 < t < t_1$ takes the form

$$Ae^{-i\omega_1 t + i\vec{k}\vec{x}}\chi_1^+ + Be^{-i(-\omega_1)t + i\vec{k}\vec{x}}\chi_1^- \quad \text{with } A, B \in \mathbb{R}.$$

Calculate A and B for the case $k_1 = 1$ and $V = m$ by demanding continuity of the solution at $t = t_0$.

(vi) Assume that for $t > t_1$ the solution takes the form

$$Ce^{-i\omega_0 t + i\vec{k}\vec{x}}\chi_0^+ + De^{-i(-\omega_0)t + i\vec{k}\vec{x}}\chi_0^- \quad \text{with } C, D \in \mathbb{C}.$$

Calculate C and D for $m = 2$ by demanding continuity of the solution at $t = t_1$ (you may want to use Mathematica here).

(vii) Interpret what you have found. Why could this be called the "external field *problem*"?