Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 15

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) What is a local foliation in the smooth setting? How can the Cauchy problem be implemented thereon?

(ii) What does it mean to soften the surface layer integrals? In which sense are they "soft"?

(iii) This softening procedure is very convenient, because it allows us to prove useful identities, such as the Green's and the energy identities. How can one prove them?

(iv) The existence theory is based on a very important assumption which gives better control on the growth rate of the solutions. Which one? How is it implemented in the proof of the existence theorem?

Exercises

Exercise 15.1: On the surface-layer inner product

The goal of this exercise to show that, under a suitable restriction of the jet space, the surface-layer inner product is indeed non-negative. On $\mathscr{F} = \mathbb{R}^2$ we define the Lagrangian

$$\mathcal{L}(x,y) = \frac{1}{2}\,\eta(x_1 - y_1)\,(x_2 - y_2)^2, \quad \text{where} \;\; \eta \in C_0^\infty(\mathbb{R}, \mathbb{R}^+).$$

Let $M = \mathbb{R} \subset \mathscr{F}$ equipped with the canonical measure one dimensional Lebesgue measure and consider the set of jets

$$\mathfrak{J} := \left\{ (0,u) \;\middle|\; u = \sum_{i=1}^{2} u_i \partial_i \in T\mathscr{F} \text{ with } u_1(t,0) = 0 \text{ and } \partial_1 u_2(t,0) \leq 0 \text{ for all } t \in \mathbb{R} \right\}.$$

Let $\Omega_t := (-\infty, t) \subset M$. Show that the surface-layer inner product $(\,\cdot\,,\,\cdot\,)^{\Omega_t}|_{\mathfrak{J}\times\mathfrak{J}}$ is non-negative.

Hint: Remember that jets are never differentiated in expressions like $\nabla_{i,\mathfrak{v}}\nabla_{j,\mathfrak{u}}$.

Exercise 15.2: Linearized fields on the sphere

Let ρ be a minimizing measure of the causal variational principle of the sphere.

(i) Let v be the vector field $\partial/\partial\varphi$ (where φ is the azimuth angle). Show that $\mathfrak{v} = (0, v)$ is a solution of the linearized field equations. *Hint:* One can use the fact that the causal variational principle is rotationally symmetric.

(ii) Show that $\mathfrak{v}$ can be written as a commutator jet, i.e. in analogy to (**??**),

$$v(x) = i \left[c\sigma^3, F(x) \right] ,$$

where $F : S^2 \subset \mathbb{R}^3 \to \mathscr{F}$ is the mapping in (**??**). Compute the constant c.

Exercise 15.3: Linearized fields for the causal variational principle on $\mathbb{R}$

We return to the causal variational principles on $\mathbb{R}$ introduced in Exercise 13.5. Let $\rho = \delta$ be the unique minimizer.

(i) Show that the jet $\mathfrak{v} = (0, v)$ with the vector field $v = \partial_x$ is a solution of the linearized field equations for the causal variational principle corresponding to $\mathcal{L}_4$.

(ii) Show that the jet $\mathfrak{v} = (0, v)$ from (i) does *not* satisfy the linearized field equations for the causal variational principle corresponding to $\mathcal{L}_2$.

Exercise 15.4: Linearized fields for the causal variational principle on S^1

We return to the causal variational principle on $\mathbb{R}$ introduced in Exercise 13.6. Let ρ be a minimizing measure for $0 < \tau < 1$.

(i) Show that the jet $\mathfrak{v} = (0, v)$ with the vector field $v = \partial_\varphi$ satisifies the linearized field equations. *Hint:* One can use the fact that the variational principle is rotationally symmetric.

(ii) Show that the jet $\mathfrak{v} = (b, 0)$ with $b(\phi_0) = -b(\phi_0 + \pi)$ is a solution of the linearized field equations. *Hint:* Use that the causal action is independent of the parameter τ.

(iii) Show that every solution of the linearized field equations is a linear combination of the linearized fields in (i) and (ii).