Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 14

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) How can non-compactness in the setting of causal fermion systems of finite dimensions be overcome? What is a moment measure? What does Lebesgue's decomposition theorem say?

(ii) What is a variation of finite volume? What a compact variation? By means of what approximation method can the existence of minimizers in the compact setting be used to prove existence on non-compact manifolds?

Exercises

Exercise 14.1: Derivative of measures

Let μ be the counting measure on the σ-algebra $\mathcal{P}(\mathbb{N})$. Consider the measure

$$\lambda(\emptyset) := 0, \quad \lambda(E) := \sum_{n \in E} (1+n)^2, \quad E \in \mathcal{P}(\mathbb{N}).$$

Show that μ and λ are equivalent (one is absolutely continuous with respect to the other) and determine the Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$.

Exercise 14.2: Minimizers

Let M denote the 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ and let $d\mu_M$ be the normalized canonical surface measure. Consider a Lagrangian on $M \times M$ defined by

$$\mathcal{L}(x,y) := \frac{1}{1 + \|x-y\|_{\mathbb{R}^3}} \quad \text{for all } x, y \in M.$$

Show that the action $\mathcal{S}(\mu_M)$ is minimal under variations of the form

$$d\rho_{x_0,t} := (1-t)\,d\mu_M + t\,d\delta_{x_0}, \quad \text{with } t \in [0,1),$$

where δ_{x_0} is the Dirac measure centered at $x_0 \in M$.

Exercise 14.3: Moment measures

Let $\mathscr{F} = \mathbb{R}^2$ and $K = \mathbb{S}^1 \cup \{0\}$ be a compact subset of $\mathscr{F}$. Given a Borel measure ρ on $\mathscr{F}$, the moment measures $\mathfrak{m}^{(0)}, \mathfrak{m}^{(1)}$ and $\mathfrak{m}^{(2)}$ can be defined just as in the lecture. Compute these moment measures for the following choices of ρ:

(i) $\rho = F_*(\mu_{\mathbb{S}^1})$, where $F : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ is the natural injection and $\mu_{\mathbb{S}^1}$ is the normalized Lebesgue measure on $\mathbb{S}^1$.

(ii) $\rho = \delta_{(0,0)} + \delta_{(1,1)} + \delta_{(5,0)}$ (where $\delta_{(x,y)}$ denote the Dirac measure supported at $(x, y) \in \mathbb{R}^2$).

(iii) $\rho = F_*(\mu_{\mathbb{R}})$, where $\mu_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$ and

$$F : \mathbb{R} \to \mathbb{R}^2, \quad F(x) = (x, 2).$$

Exercise 14.4: Linearization of nonlinear partial differential equations

In this exercise you are given two *non-linear* partial differential equations with corresponding (soliton) solutions. Check that the functions ϕ do indeed solve the equations. Then try to figure out what it means to *linearize* the equations arounds the given solutions and do it.

(i) The *sine-Gordon equation* of velocity $v \in (-1, 1)$:

$$\partial_{tt}\phi - \partial_{xx}\phi + \sin\phi = 0, \quad \phi(t, x) = 4 \arctan\left(e^{\frac{x-vt}{\sqrt{1-v^2}}}\right).$$

(ii) The *Korteweg-de-Vries equation* of unit speed:

$$\partial_t\phi + 6\,\phi\,\partial_x\,\phi + \partial_{xxx}\phi = 0, \quad \phi(t, x) = \frac{1}{2}\operatorname{sech}^2\left(\frac{x - vt}{2}\right).$$

Hint: You may use the following identities

$$\sin(4\arctan(x)) = -4\frac{x^3 - x}{(1 + x^2)^2}, \quad \tanh(x) - \tanh^3(x) = \operatorname{sech}^2(x)\tanh(x).$$