Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 13

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) The existence of minimizers in the compact setting is based upon two important results of functional analysis. What are they?

(ii) How it the existence of minimizers proved? What is the direct method in the calculus of variations?

(iii) How are the two results be used to prove existence in the compact setting? Why is compactness essential for the method to work?

Exercises

Exercise 13.1: Riesz representation theorem - part 1

Let Λ be the functional

$$\Lambda : C^0([0,1],\mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) := \sup_{x\in[0,1]} f(x).$$

Can this functional be represented by a measure? Analyze how your findings are compatible with the Riesz representation theorem.

Exercise 13.2: Riesz representation theorem - part 2

Let $\mathcal{M}$ be a closed embedded submanifold of $\mathbb{R}^3$. We choose a compact set $K \subset \mathbb{R}^3$ which contain $\mathcal{M}$. On $C^0(K,\mathbb{R})$ we introduce the functional

$$\Lambda : C^0(K,\mathbb{R}) \to \mathbb{R}, \quad \Lambda(f) = \int_{\mathcal{M}} f(x)\, d\mu_{\mathcal{M}}(x),$$

where $d\mu_{\mathcal{M}}$ is the volume measure corresponding to the induced Riemannian metric on $\mathcal{M}$. Show that this functional is linear, bounded and positive. Apply the Riesz representation theorem to represent this functional by a Borel measure on K. What is the support of this measure?

Exercise 13.3: Lebesgue decomposition theorem

Let ρ be the Borel measure on $[0, \pi]$ given by

$$\rho(\Omega) := \int_\Omega \sin x \, dx + \sum_{n=1}^\infty \frac{1}{n^2} \, \chi_\Omega\Big(\frac{1}{n}\Big) \, .$$

Compute the Lebesgue decomposition of ρ with respect to the Lebesgue measure.

Exercise 13.4: Normalized regular Borel mesures: compactness results

(i) Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of normalized regular Borel measures on $\mathbb{R}$ with the property that there is a constant $c > 0$ such that

$$\int_\mathbb{R} x^2 \, d\rho_n(x) \leq c \quad \text{for all } n \in \mathbb{N}.$$

Show that a subsequence converges again to a normalized Borel measure on $\mathbb{R}$.
Hint: Apply the compactness result of the lecture to the measures restricted to the interval $[-L, L]$ *and analyze the behavior as* $L \to \infty$.

(ii) More generally, assume that for a given non-negative function $f \in C^0(\mathbb{R}, \mathbb{R})$,

$$\int_\mathbb{R} f(x) \, d\rho_n(x) \leq c \quad \text{for all } n \in \mathbb{N}.$$

Which condition on f ensures that a subsequence of the measures converges to a normalized Borel measure? Justify your results by a counterexample.

Exercise 13.5: A causal variational principle on $\mathbb{R}$

We let $\mathscr{F} = \mathbb{R}$ and consider the Lagrangians

$$\mathcal{L}_2(x, y) = (1 + x^2)(1 + y^2) \qquad \text{and} \qquad \mathcal{L}_4(x, y) = (1 + x^4)(1 + y^4) \, .$$

We minimize the corresponding causal actions within the class of all normalized regular Borel measures on $\mathbb{R}$. Show with a direct estimate that the Dirac measure δ supported at the origin is the unique minimizer of these causal variational principles.

Exercise 13.6: A causal variational principle on S^1

We let $\mathscr{F} = S^1$ be the unit circle parametrized as $e^{i\varphi}$ with $\varphi \in \mathbb{R} \bmod 2\pi$ and consider the Lagrangian

$$\mathcal{L}(\varphi, \varphi') = 1 + \sin^2 \big(\varphi - \varphi' \bmod 2\pi\big) \, .$$

We minimize the corresponding causal action within the class of all normalized regular Borel measures on S^1. Show by direct computation and estimates that every minimizer is of the form

$$\rho = \tau \, \delta\big(\varphi - \varphi' - \varphi_0 \bmod 2\pi\big) + (1 - \tau) \, \delta\big(\varphi - \varphi' - \varphi_0 + \pi \bmod 2\pi\big)$$

for suitable values of the parameters $\tau \in [0, 1]$ and $\varphi_0 \in \mathbb{R} \bmod 2\pi$.