Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 12

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) How does the conserved one-form arise from the general class of conservation laws?

(ii) In general, the conserved one-form does not provide the conservation of a surface layer integral. Why so? For which class of jets is this true instead?

(iii) Restricting attention to inner solutions, the volume term in the conserved one-form can be analyzed in more detail. How can it be understood mathematically?

(iv) Restricting attention to commutator jets, the conserved one-form gives rise to an additional canonical inner product on the Hilbert space. How is that exactly?

(v) How does the symplectic form arise from the general class of conservation laws? It is defined by means of an anti-symmetrization process. Does a symmetrization lead to anything useful?

(vi) What is a nonlinear conservation law? How can be the conserved one-form and the surface layer inner product be retrived from the nonlinear surface layer integral?

(vii) How can the area of a two-dimensional surface be defined?

Exercises

Exercise 12.1: On inner solutions

Let $d\rho \in \Omega^n M$ be a volume form and consider differentiable one-parameter families $h_t \in C^\infty(M, \mathbb{R}^+)$, $\Phi_t \in \mathrm{Diff}(M)$, with the property that

$$\dot{h}_0 = 0, \quad \Phi_0 = \mathrm{Id}_M, \quad h_0\, d\rho = (\Phi_t)^*(h_t\, d\rho) \quad \text{for all } t \in \mathbb{R}. \tag{1}$$

Rewrite (1) as in Exercise 8.4 (ii) and show that the corresponding inner solution depens only on h_0 and X_0.

Exercise 12.2: Moser's theorem

Let M be an connected, oriented and closed manifold of dimension n. Let $\omega_0, \omega_1 \in \Omega^n M$ be volume forms with the property that

$$\int_M \omega_0 = \int_M \omega_1 .$$

Find a diffeomorphism $\Phi \in \mathrm{Diff}(M)$ such that $\omega_0 = \Phi^* \omega_1$.

Hint: First, apply Exercise 8.2. Find then a simple differentiable curve ω_s of volume forms connecting ω_0 and ω_1 and apply Exercise 8.3, obtaining in this way a time-dependent vector field. Using Exercise 8.4 show that the pull-back of ω_s through the flow is constant. What is then Φ?

Exercise 12.3: Nonlinear surface layer integral

We now use Moser's theorem to prove conservation of the nonlinear surface layer integral in the compact setting. To this aim, let $\mathscr{F}$ be a smooth manifold and $\mathcal{L} \in C^\infty(\mathscr{F} \times \mathscr{F}, \mathbb{R}^+)$. Furthermore, let M_0, M_1 be two connected, oriented and closed submanifolds of $\mathscr{F}$ of dimension n, and let $F \in \text{Diff}(M_0, M_1)$. Finally, let $d\rho_0 \in \Omega^n M_0$ and $d\rho_1 \in \Omega^n M_1$ be two volume forms. The *nonlinear surface layer integral* is defined for any relatively compact $\Omega \subset M_0$ by

$$\gamma_F^\Omega(\rho_0, \rho_1) := \int_{F(\Omega)} d\rho_1(x) \int_{M_0 \setminus \Omega} d\rho_0(y)\, \mathcal{L}(x,y) - \int_\Omega d\rho_0(x) \int_{M_1 \setminus F(\Omega)} d\rho_1(y)\, \mathcal{L}(x,y),$$

where ρ_0 and ρ_1 are the measures induced by the volume forms on the correponding manifolds. The goal of this exercise is the following: *Show that F can be modified in a way that $\gamma_F^\Omega(\rho_0, \rho_1) = 0$ for all relatively compact $\Omega \subset M_0$.* To this aim, let us introduce the volume forms

$$d\nu_0(x) := \left(\int_{M_1} \mathcal{L}(x,y)\, d\rho_1(y) \right) d\rho_0(x) \quad \text{and} \quad d\nu_1(x) := \left(\int_{M_0} \mathcal{L}(x,y)\, d\rho_0(y) \right) d\rho_1(x)$$

(i) Using the symmetry of the Lagrangian $\mathcal{L}(x,y) = \mathcal{L}(y,x)$ show that, for all Ω,

$$\gamma_F^\Omega(\rho_0, \rho_1) = \int_\Omega (F^* d\nu_1 - d\nu_0).$$

(ii) Using Exercise 9.1 show that there is $\Phi \in \text{Diff}(M_0)$ such that $(F \circ \Phi^{-1})^* d\nu_1 = d\nu_0$.

Hint: Note that M_0 and M_1 are different manifolds! How do you apply Exercise 9.1?

Points (i) and (ii) together give the claim: By modifying F to $F \circ \Phi^{-1}$ we obtain a conservation law for the nonlinear surface layer integral.

Exercise 12.4: Commutator jets and conserved surface layer integrals

Let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system on a finite-dimensional Hilbert space. For any self-adjoint $S \in L(\mathscr{H})$, we define the corresponding *commutator jet* by

$$\mathfrak{C}_S := (0, \mathcal{C}_S), \quad \text{with} \quad \mathcal{C}_S(x) := i[S, x] \quad \text{for all } x \in \mathscr{F}.$$

Prove the following identity between the conserved one-form and the conserved symplectic form:

$$\gamma_\rho^\Omega\big((0, [\mathcal{C}_A, \mathcal{C}_B])\big) = -\frac{1}{2}\, \sigma_\rho^\Omega(\mathfrak{C}_A, \mathfrak{C}_B),$$

where $[\mathcal{C}_A, \mathcal{C}_B]$ denotes the commutator of vector fields on $\mathscr{F}$.