Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 11

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) What kind of infinitesimal variations do commutator jets and inner solutions describe?

(ii) What is a surface layer integral? In which sense does it generalize the concept of a surface integral? In which limiting case can the latter be recovered?

(iii) How does a symmetry of the Lagrangian correspond to the conservation of a specific surface layer integral?

(iv) How does the general class of conserved surface layer integrals look like?

Exercises

Exercise 11.1: Noether-like theorems

The goal of this exercise is to illustrate the Noether-like theorems mentioned in the lecture. In order to simplify the problem as far as possible, we consider the compact setting and assume furthermore that the Lagrangian is smooth, i.e. $\mathcal{L} \in C^\infty(\mathscr{F} \times \mathscr{F}, \mathbb{R}^+_0)$. Let ρ be a minimizer of the action under variations of ρ in the class of (positive) normalized regular Borel measures. Let $u \in T\mathscr{F}$ be a vector field on $\mathscr{F}$. Assume that u is a symmetry of the Lagrangian in the sense that

$$\left(u(x)^j \frac{\partial}{\partial x^j} + u(y)^j \frac{\partial}{\partial y^j} \right) \mathcal{L}(x,y) = 0 \quad \text{for all } x, y \in \mathscr{F}. \tag{1}$$

Prove that for any measurable set $\Omega \subset \mathscr{F}$,

$$\int_\Omega d\rho(x) \int_{\mathscr{F} \setminus \Omega} d\rho(y)\, u(x)^j \frac{\partial}{\partial x^j} \mathcal{L}(x,y) = 0.$$

Hint: Integrate (1) *over* $\Omega \times \Omega$*. Transform the integral using the symmetry* $\mathcal{L}(x,y) = \mathcal{L}(y,x)$*. Finally make use of the Euler-Lagrange equations.*

Exercise 11.2: In preparation for Moser Theorem - part 1

Let M be a connected, oriented and closed manifold of dimension n and $\omega \in \Omega^n M$ a volume form. It can be proved that

$$\int_M \omega = 0 \quad \text{if and only if there exists } \eta \in \Omega^{n-1} M \text{ such that } \omega = d\eta.$$

One direction follows immediately from Stokes Theorem. The goal of this exercise is to understand the idea behind the proof of the other direction by working in a very simple setting. More precisely, you are asked to prove the following statement: Let $n = 1, 2$ and $\omega \in \Omega^n\mathbb{R}^n$ a differentiable form with compact support contained in $(0,1)^n$. Show that there exists a differential form $\eta \in \Omega^{n-1}\mathbb{R}^n$, again with compact support in $(0,1)^n$, such that $\omega = d\eta$.

Hint: The case $n = 1$ is immediate. The case $n = 2$ is a bit trickier. The proof boils down to finding a smooth function $F \in C_0^\infty((0,1)^2, \mathbb{R}^2)$ such that $\partial_1 F^2 - \partial_2 F^1 = \omega(\partial_1, \partial_2)$. Try to reduce the problem to the one-dimensional case by treating one variable as a parameter. Taking an average over $(0,1)$ may be useful.

Exercise 11.3: In preparation for Moser Theorem - part 2

Let M be a n-dimensional manifold and $\omega \in \Omega^n M$ a volume form. It can be proved that for any $\eta \in \Omega^{n-1}M$ there exists a unique vector field $X \in TM$ such that $\imath_X\omega = \eta$. Once again, the goal of this exercise is to understand the proof by working in very specific setting. More precisely, you are asked to prove the statement in the case $n = 2$.

Hint: How could the Fréchet-Riesz representation theorem be exploited here? Differentiability may be proved by working with local trivializations.

Exercise 11.4: In preparation for Moser theorem - part 3

Let M be an n-dimensional compact manifold. A *time-dependent vector field* is a differentiable function $X : \mathbb{R} \times M \to TM$ such that $X(t,p) \in T_pM$ for all $t \in \mathbb{R}$ and $p \in M$. Its *flow* is the unique differentiable mapping $\alpha : \mathbb{R} \times M \to M$ with the property that $\alpha(\,\cdot\,, p)$ is the integral curve of X with initial condition p, i.e. $\alpha(0,p) = p$ and $\alpha'(t,p) = X(t, \alpha(t,p))$ for all $t \in \mathbb{R}$. We use the simpler notation $X_t := X(t, \cdot)$ and $\alpha_t := \alpha(t, \cdot)$.

(i) It can be proved that, for any differential form $\omega \in \Omega^k M$,

$$\left.\frac{d}{ds}\right|_t (\alpha_s^*\omega) = \alpha_t^*(\mathcal{L}_{X_t}\omega), \tag{2}$$

where $\mathcal{L}_\xi$ is the *Lie derivative* along the vector field ξ and * denotes the pull-back. Check identity (2) for ω of the form f, dh and $f dh$, where $f, h \in C^\infty(M)$.

Hint: How does X_t act on f? Look up the definition and the properties of the pull-back and the Lie derivative. In particular, remember that d commute with *, $\mathcal{L}_\xi$ and d/ds.

(ii) Let $d\rho \in \Omega^n M$ be a volume form. Assume the existence of two differentiable families

$$\Phi : \mathbb{R} \to \mathrm{Diff}(M) \text{ and } \varphi : \mathbb{R} \to C^\infty(M), \text{ with } \Phi_0 = \mathrm{Id}_M \text{ and } \varphi_0 \equiv 1,$$

such that

$$d\rho = (\Phi_t)^*(\varphi_t\, d\rho) \quad \text{for all } t \in \mathbb{R}. \tag{3}$$

We now introduce the *time-dependent vector field*

$$X(t,p) := \left.\frac{d}{ds}\right|_t \Phi_s(\Phi_t^{-1}(p)) \in T_pM.$$

Show that Φ is the flow of X and that the couple $(-\dot{\varphi}_0, X_0)$ defines an *inner solution.*

Hint: Differentiate (3) through using the chain rule. What is the connection between the divergence and the Lie derivative?