Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 10

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the lecture of the present week and help you through the literature.

(i) What are the Euler-Lagrange equations? What are they good for?

(ii) In which setting are the EL equations derived? Do they hold in the general setting of causal fermion systems? Under which additional assumptions?

(iii) What is a jet? How do the linearized field equations follow from the EL equations? In which settings can they be implemented?

Exercises

Exercise 10.1: A distinguished Riemannian metric on the regular points (2 points)

Let $\mathscr{H}$ be a Hilbert space of finite dimension N and let $n < N/2$ denote the spin dimension. The set of regular points $\mathscr{F}^{\text{reg}}$ inherits from the ambient space $L(\mathscr{H}) \cong \mathbb{R}^{2(N \times N)}$ a smooth manifold structure of dimension $4n(N-n)$. For any $x \in \mathscr{F}^{\text{reg}}$ we introduce the bilinear form

$$h_x : T_x\mathscr{F}^{\text{reg}} \times T_x\mathscr{F}^{\text{reg}} \ni (u, v) \mapsto \text{tr}(uv) \in \mathbb{R}. \tag{1}$$

The goal of this exercise is to show that h is a well-defined *Riemannian metric* on $\mathscr{F}^{\text{reg}}$.

(i) Show that every $u \in T_x\mathscr{F}^{\text{reg}} \subset L(\mathscr{H})$ is self-adjoint.

(ii) Show that (1) is a positive-definite inner product which depends smoothly on x.

Exercise 10.2: Variations of the universal measure and the physical wave functions

Let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system with $M := \text{supp}\, \rho$. Let $F_\tau : M \to \mathscr{F}$ with $\tau \in (-\delta, \delta)$ be a family of continuous functions which satisfy the following conditions,

$$F_0 = \mathbb{I} \quad \text{and} \quad F_\tau|_{M \setminus K} \equiv \mathbb{I} \quad \text{with } K \subset M \text{ compact.}$$

Moreover, we assume that F_τ is differentiable in τ and that all points of K are regular. This gives rise to a variation of the universal measure by defining $\rho_\tau := (F_\tau)_* \rho$. This exercise explains how such a variation can be realized by a variation of the physical wave functions.

(i) Fix $x \in K$. Show that by decreasing δ, one can arrange that the operators $F_\tau(x)$ have maximal rank $2n$ for all $\tau \in (-\delta, \delta)$.

(ii) We introduce the spin spaces

$$S_x^\tau := \operatorname{Im} F_\tau(x), \quad \prec \cdot \,|\, \cdot \succ_x^\tau := -\langle \cdot \,| F_\tau(x) \, \cdot \rangle_{\mathscr{H}} |_{S_x^\tau \times S_x^\tau}.$$

Construct a family of isometries (w.r.t. the spin inner products)

$$V_\tau(x) : (S_x, \prec \cdot \,|\, \cdot \succ_x) \to (S_x^\tau, \prec \cdot \,|\, \cdot \succ_x^\tau)$$

which is differentiable in τ. *Hint: For example, one can work with the orthogonal projections in $\mathscr{H}$ and take the polar decomposition with respect to the spin scalar products.*

(iii) Consider the variation of the wave evaluation operator given by

$$\Psi_\tau(x) := V_\tau(x)^{-1} \pi_{F_\tau(x)} : \mathscr{H} \to S_x.$$

Show that $F_\tau(x) = -\Psi_\tau(x)^* \Psi_\tau(x)$.

(iv) So far, the point $x \in K$ was fixed. We now extend the construction so that x can be varied. Use a compactness argument to show that there is $\delta > 0$ such that the operators $F_\tau(x)$ have maximal rank $2n$ for all $\tau \in (-\delta, \delta)$ and all $x \in K$. Show that the mappings $V_\tau(x)$ can be introduced such that they depend continuously in x and are differentiable in τ.

Exercise 10.3: Unitary variations in the Krein space

Let $(\mathscr{H}, \mathscr{F}, \rho)$ be a causal fermion system with corresponding Krein space $(\mathscr{K}, <\cdot|\cdot>)$. Assume that $\psi^u \in \mathscr{K}$ for all $u \in \mathscr{H}$, so that the evaluation operator $\Psi : \mathscr{H} \to \mathscr{K}$ is well-defined. Let U_τ be a continuous family of unitary operators on $\mathscr{K}$. Setting

$$\Psi_\tau := U_\tau \circ \Psi : \mathscr{H} \to \mathscr{K},$$

we obtain a corresponding variation of the physical wave functions, which gives rise to a variation of the universal measure by setting

$$\rho_\tau := (F_\tau)_* \rho \quad \text{with} \quad F_\tau(x) := -\Psi_\tau(x)^* \Psi_\tau(x).$$

Show that the volume and the trace constraints are satisfied by these variations.

Exercise 10.4: Jets in Minkowski Space

Let $\mathscr{H}_m^-$ the Hilbert space of negative-energy solutions of the Dirac equation. We introduce a *regularization operator* by inserting a cutoff in momentum space. More precisely, let χ denote the characteristic function of the unit ball in $\mathbb{R}^3$. Then, on the dense subset $\mathcal{D} \subset \mathscr{H}_m^-$ of smooth solutions u of the form (cf. Exercise 1.2)

$$u(x) := \int_{\mathbb{R}^3} d^3\boldsymbol{k}\, p_-(\boldsymbol{k}) \varphi(\boldsymbol{k})\, e^{i(\omega(\boldsymbol{k})t + \boldsymbol{k}\cdot\boldsymbol{x})}, \quad \text{with } \varphi \in S(\mathbb{R}^3, \mathbb{C}^4), \tag{2}$$

we define $\mathfrak{R}_\varepsilon u \in \mathscr{H}_m^-$ by cutting off the momenta larger than ε^{-1}, i.e.

$$(\mathfrak{R}_\varepsilon u)(x) := \int_{\mathbb{R}^3} d^3\boldsymbol{k}\, \chi(\varepsilon \boldsymbol{k})\, p_-(\boldsymbol{k}) \varphi(\boldsymbol{k})\, e^{i(\omega(\boldsymbol{k})t + \boldsymbol{k}\cdot\boldsymbol{x})}.$$

(i) Show that $\mathfrak{R}_\varepsilon : \mathcal{D} \to \mathscr{H}_m^-$ is continuous, hence it can be uniquely extended to all of $\mathscr{H}_m^-$. Show that such extension is self-adjoint and idempotent.

(ii) Define $\mathscr{H}^\varepsilon := \operatorname{Im} \mathfrak{R}_\varepsilon$ and $F^\varepsilon : \mathbb{R}^4 \to \mathscr{F}$ as in Exercise 6.2-(1). Show that $\operatorname{Im} F^\varepsilon(x) \subset \mathscr{H}^\varepsilon$.

For any $a \in \mathbb{R}^4$ we define the *translation operators*

$$U_a : \mathscr{H}_m^- \to \mathscr{H}_m^-, \quad (U_a u)(x) := u(x+a).$$

(iii) Working in momentum space with solutions of the form (2) show that U is well-defined and that it gives rise to a strongly-continuous (i.e. $U_a u \to u$ as $a \to 0$ for all u) unitary representation of the translation group $\mathbb{R}^4$

(iv) Show that $U_a(\mathscr{H}^\varepsilon) \subset \mathscr{H}^\varepsilon$ and that the self-adjoint generators of U are bounded on $\mathscr{H}^\varepsilon$, i.e. show that

$$\mathscr{H}^\varepsilon \ni u \mapsto i \frac{d}{ds}\bigg|_0 U_{se_j} u \in \mathscr{H}^\varepsilon$$

is a well-defined bounded operator for any $j = 0, 1, 2, 3$.

Hint: Work on the dense subset $\mathfrak{R}_\varepsilon(\mathcal{D}) \subset \mathscr{H}^\varepsilon$

(v) Show that $[U_a, \mathfrak{R}_\varepsilon] = 0$ and $U_a^\dagger\, F^\varepsilon(x)\, U_a = F^\varepsilon(x+a)$.

(vi) In view of the considerations above, we can focus our attention to the cutoff subspace $\mathscr{H}^\varepsilon$. Consider the differentiable families of transformations ($M^\varepsilon := \operatorname{supp}(F^\varepsilon)_*(d^4x)$)

$$\Phi_j : M^\varepsilon \times \mathbb{R}^4 \ni (A, s) \mapsto U_{se_j}^\dagger\, A\, U_{se_j} \in M^\varepsilon.$$

These transformation can be described infinitesimally by jets $\mathfrak{u}_j = (0, u_j)$. Determine u_j and show that they solve the weak Euler-Lagrange equations.