Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 1

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) Causality in Minkowski space: What is the physical meaning of the causal structure? How is it described mathematically? What is the light cone?

(ii) Dirac and Klein-Gordon equations: What are the conservation laws? What does conservation mean? How are the conservation laws verified?

Exercises

Remember the following identities, where * and † denote the adjoint with respect to the spin inner product and the standard scalar product on $\mathbb{C}^4$, respectively:

$$(\gamma^i)^* = \gamma^i \quad \text{and} \quad (\gamma^i)^\dagger = \eta^{ii}\gamma^i \quad \text{for all } i = 1, 2, 3, 4.$$

Exercise 1.1: Positive and negative energy splitting

Let us define for every $\boldsymbol{k} \in \mathbb{R}^3$ the energy $\omega(\boldsymbol{k}) := \sqrt{\boldsymbol{k}^2 + m^2}$ and the matrices

$$p_\pm(\boldsymbol{k}) := \frac{\not{k} + m}{2\,k^0}\,\gamma^0\bigg|_{k^0 = \pm\omega(\boldsymbol{k})} \in \mathrm{Mat}(4, \mathbb{C}) \quad (\text{with } \not{k} := k^0\gamma^0 - \boldsymbol{k}\cdot\boldsymbol{\gamma}).$$

(i) Referring to the standard scalar product of $\mathbb{C}^4$, show that the matrices $p_\pm(\boldsymbol{k})$ are symmetric, idempotent, add up to the identity and have orthogonal images. Conclude that the spinor space $\mathbb{C}^4$ can be decomposed into the orthogonal direct sum

$$\mathbb{C}^4 = W^+_{\boldsymbol{k}} \oplus W^-_{\boldsymbol{k}}, \quad \text{with} \quad W^\pm_{\boldsymbol{k}} := \mathrm{Im}\, p_\pm(\boldsymbol{k}).$$

(ii) Let $\varphi \in C^\infty_{\mathrm{sc}}(\mathbb{R}^4, \mathbb{C}^4)$ be a smooth solution of the Dirac equation with spatially compact support, i.e.

$$(i\gamma^0\partial_0 + i\gamma^\alpha\partial_\alpha - m)\varphi = 0, \quad \text{with} \;\; \varphi(t, \cdot\,) \in C^\infty_0(\mathbb{R}^3, \mathbb{C}^4) \text{ for all } t \in \mathbb{R}.$$

Let $\hat{\varphi}$ be the smooth function on $\mathbb{R}^4$ defined by taking the Fourier transform of φ in the spatial variables only. Find $h \in C^\infty(\mathbb{R}^3, \mathrm{Mat}(4, \mathbb{C}))$ such that

$$i\partial_t\, \hat{\varphi}(t, \boldsymbol{k}) = h(\boldsymbol{k}) \cdot \hat{\varphi}(t, \boldsymbol{k}) \quad \text{for all } t \in \mathbb{R},\; \boldsymbol{k} \in \mathbb{R}^3.$$

Show that $h(\boldsymbol{k})$ is also symmetric with respect to the standard scalar product of $\mathbb{C}^4$ and satisfies

$$h(\boldsymbol{k})p_\pm(\boldsymbol{k}) = \pm\omega(\boldsymbol{k})p_\pm(\boldsymbol{k}).$$

In particular, $\pm\omega(\boldsymbol{k})$ form the spectrum of $h(\boldsymbol{k})$.

(iii) Referring to point *ii.*), conclude that

$$\varphi(t,\boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \big(p_-(\boldsymbol{k})\hat{\varphi}(0,\boldsymbol{k})\, e^{i\omega(\boldsymbol{k})t} + p_+(\boldsymbol{k})\hat{\varphi}(0,\boldsymbol{k})\, e^{-i\omega(\boldsymbol{k})t}\big) e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Hint: Remember that the Cauchy problem admits unique smooth solutions.

From a mathematical point of view, the *Dirac Sea* is described by the Hilbert space generated by all the smooth solutions with spatially compact support and the property that $p_-(\boldsymbol{k})\hat{\varphi}(0,\boldsymbol{k}) = \hat{\varphi}(0,\boldsymbol{k})$.

Exercise 1.2: Dirac equation in presence of external fields

In presence of an external electromagnetic field, the Dirac equation is modified to

$$(i\gamma^j(\partial_j - iA_j) - m)\psi = 0, \quad A_j \in C^\infty(\mathbb{R}^4, \mathbb{C}^4).$$

Show that, multiplying by the operator $(i\gamma^j(\partial_j - iA_j) + m)$ and using the anti-commutation relations, the following equation holds:

$$\Big(-\eta^{kl}(\partial_k - iA_k)(\partial_l - iA_l) + \frac{i}{2}\, F_{jk}\gamma^j\gamma^k - m^2\Big)\psi = 0\,, \text{ with } \; F_{jk} := \partial_j A_k - \partial_k A_j.$$

This differs from the Klein-Gordon equation by the extra term $\frac{i}{2}F_{jk}\gamma^j\gamma^k$, which describes the coupling of the spin to the electromagnetic field.

Exercise 1.3: Causality in the setting of symmetric hyperbolic systems

The Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$ can be rewritten as a *symmetric hyperbolic system*, i.e. in the form $(c > 0)$

$$(A^0(x)\,\partial_0 + A^\alpha(x)\,\partial_\alpha + B(x))\psi = 0, \quad \text{with } \; (A^i)^\dagger = A^i \; \text{ and } \; A^0(x) \geq c\mathbb{I}.$$

For such systems a notion of *causality* can be introduced: a vector $\xi \in \mathbb{R}^4$ is said to be *time-like* or *light-like* at $x \in \mathbb{R}^4$, if the matrix $A(x,\xi) := A^i(x)\,\xi_i$ is definite (either positive or negative) or singular, respectively.

Find the matrices A^i and B for the Dirac equation and show that the above notions of time-like and light-like vectors coincide with the corresponding notions in Minkowski space as explained in the lecture.