Online Course on Causal Fermion Systems

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Guiding Questions and Exercises 1

Guiding Questions

The purpose of the following questions is to highlight the main topics covered in the online course and help you through the literature.

(i) Causality in Minkowski space: What is the physical meaning of the causal structure? How is it described mathematically? What is the light cone?

(ii) Dirac and Klein-Gordon equations: What are the conservation laws? What does conservation mean? How are the conservation laws verified?

Exercises

Remember the following identities, where * and † denote the adjoint with respect to the spin inner product and the standard scalar product on $\mathbb{C}^4$, respectively:

$$(\gamma^i)^* = \gamma^i \quad \text{and} \quad (\gamma^i)^\dagger = \eta^{ii}\gamma^i \quad \text{for all } i = 0, \ldots, 3$$

(and $\eta = \mathrm{diag}(1, -1, -1, -1)$ denotes the Minkowski metric).

Exercise 1.1: Anti-commutation relations

(a) Verify by direct computation that the Dirac matrices satisfy the anti-commutation relations.

(b) Why is it not possible to satisfy these anti-commutation relations with 2×2 or 3×3-matrices? *Hints:* The case of odd-dimensional matrices can be ruled out by computing the square and the trace of the matrix $\gamma^0\gamma^1$. For 2×2-matrices, a similar argument shows that the matrix $\gamma^0\gamma^1$ is diagonalizable, making it possible to proceed in an eigenvector basis.

Exercise 1.2:

Show that the Dirac matrices in the Dirac representation are symmetric with respect to the spin inner product. Show that this symmetry property is equivalent to the statement that the matrices $\gamma^0\gamma^j$ are Hermitian.

Exercise 1.3:

In this exercise, we shall verify that for any non-zero spinor ψ, the corresponding Dirac current vector $J^k = \prec\!\psi|\gamma^k\psi\!\succ$ is non-spacelike.

(a) Show that the matrix $\gamma^0\gamma^1$ is Hermitian and has eigenvalues ± 1. Deduce that

$$\langle \psi, \gamma^0\gamma^1\psi \rangle_{\mathbb{C}^4} \leq \|\psi\|^2_{\mathbb{C}^4} \,.$$

(b) Show that the last inequality implies that $|J^1| \leq J^0$.

(c) Use the rotational symmetry of the Dirac equation to conclude that $J^0 \geq |\vec{J}|$ (where $\vec{J} = (J^1, J^2, J^3) \in \mathbb{R}^3$).

Exercise 1.4: Positive and negative energy splitting

Let us define for every $\boldsymbol{k} \in \mathbb{R}^3$ the energy $\omega(\boldsymbol{k}) := \sqrt{\boldsymbol{k}^2 + m^2}$ and the matrices

$$p_\pm(\boldsymbol{k}) := \frac{\not{k} + m}{2\,k^0}\,\gamma^0\bigg|_{k^0 = \pm\omega(\boldsymbol{k})} \in \mathrm{Mat}(4, \mathbb{C}) \quad (\text{with } \not{k} := k^0\gamma^0 - \boldsymbol{k}\cdot\boldsymbol{\gamma}).$$

(i) Referring to the standard scalar product of $\mathbb{C}^4$, show that the matrices $p_\pm(\boldsymbol{k})$ are symmetric, idempotent, add up to the identity and have orthogonal images. Conclude that the spinor space $\mathbb{C}^4$ can be decomposed into the orthogonal direct sum

$$\mathbb{C}^4 = W^+_{\boldsymbol{k}} \oplus W^-_{\boldsymbol{k}}, \quad \text{with} \quad W^\pm_{\boldsymbol{k}} := \mathrm{Im}\, p_\pm(\boldsymbol{k}).$$

(ii) Let $\varphi \in C^\infty_{\mathrm{sc}}(\mathbb{R}^4, \mathbb{C}^4)$ be a smooth solution of the Dirac equation with spatially compact support, i.e.

$$(i\gamma^0\partial_0 + i\gamma^\alpha\partial_\alpha - m)\varphi = 0, \quad \text{with } \varphi(t, \cdot\,) \in C^\infty_0(\mathbb{R}^3, \mathbb{C}^4) \text{ for all } t \in \mathbb{R}.$$

Let $\hat{\varphi}$ be the smooth function on $\mathbb{R}^4$ defined by taking the Fourier transform of φ in the spatial variables only. Find $h \in C^\infty(\mathbb{R}^3, \mathrm{Mat}(4, \mathbb{C}))$ such that

$$i\partial_t\,\hat{\varphi}(t, \boldsymbol{k}) = h(\boldsymbol{k}) \cdot \hat{\varphi}(t, \boldsymbol{k}) \quad \text{for all } t \in \mathbb{R},\ \boldsymbol{k} \in \mathbb{R}^3.$$

Show that $h(\boldsymbol{k})$ is also symmetric with respect to the standard scalar product of $\mathbb{C}^4$ and satisfies

$$h(\boldsymbol{k})p_\pm(\boldsymbol{k}) = \pm\omega(\boldsymbol{k})p_\pm(\boldsymbol{k}).$$

In particular, $\pm\omega(\boldsymbol{k})$ form the spectrum of $h(\boldsymbol{k})$.

(iii) Referring to point (ii), conclude that

$$\varphi(t, \boldsymbol{x}) = \int_{\mathbb{R}^3} \frac{d^3\boldsymbol{k}}{(2\pi)^{3/2}} \Big(p_-(\boldsymbol{k})\hat{\varphi}(0, \boldsymbol{k})\, e^{i\omega(\boldsymbol{k})t} + p_+(\boldsymbol{k})\hat{\varphi}(0, \boldsymbol{k})\, e^{-i\omega(\boldsymbol{k})t}\Big)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$$

Hint: Remember that the Cauchy problem admits unique smooth solutions.

From a mathematical point of view, the *Dirac sea* is described by the Hilbert space generated by all the smooth solutions with spatially compact support and the property that $p_-(\boldsymbol{k})\hat{\varphi}(0, \boldsymbol{k}) = \hat{\varphi}(0, \boldsymbol{k})$.

Exercise 1.5: Dirac equation in presence of external fields

In presence of an external electromagnetic field, the Dirac equation is modified to

$$\big(i\gamma^j(\partial_j - iA_j) - m\big)\psi = 0 \qquad \text{with} \qquad A_j \in C^\infty(\mathbb{R}^4, \mathbb{C}^4)\,.$$

Show that, multiplying by the operator $(i\gamma^j(\partial_j - iA_j) + m)$ and using the anti-commutation relations, the following equation holds:

$$\Big(-\eta^{kl}(\partial_k - iA_k)(\partial_l - iA_l) + \frac{i}{2}\,F_{jk}\gamma^j\gamma^k - m^2\Big)\psi = 0\,, \text{ with } F_{jk} := \partial_j A_k - \partial_k A_j.$$

This differs from the Klein-Gordon equation by the extra term $\frac{i}{2} F_{jk} \gamma^j \gamma^k$, which describes the coupling of the spin to the electromagnetic field.

Exercise 1.6: Causality in the setting of symmetric hyperbolic systems

The Dirac equation $(i\partial\!\!\!/ - m)\psi = 0$ can be rewritten as a *symmetric hyperbolic system*, i.e. in the form ($c > 0$)

$$(A^0(x)\,\partial_0 + A^\alpha(x)\,\partial_\alpha + B(x))\psi = 0, \quad \text{with} \;\; (A^i)^\dagger = A^i \;\; \text{and} \;\; A^0(x) \geq c\mathbb{I}.$$

For such systems a notion of *causality* can be introduced as follows. A vector $\xi \in \mathbb{R}^4$ is said to be *time-like* or *light-like* at $x \in \mathbb{R}^4$, if the matrix $A(x, \xi) := A^i(x)\,\xi_i$ is definite (either positive or negative) or singular, respectively.

Find the matrices A^i and B for the Dirac equation and show that the above notions of time-like and light-like vectors coincide with the corresponding notions in Minkowski space as explained in the lecture. *Hint:* Don't be surprised if the naive choice $A^j = \gamma^j$ does not work.