Entanglement entropy of the ideal Fermi gas

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Introduction

Entanglement entropy (EE) is still much studied concept in condensed matter physics and quantum information theory. EE was introduced in mid 80's (and independently in early 90's) to explain Bekenstein–Hawking entropy of black holes. Here, this entropy grows with surface area rather than with volume as in usual thermodynamics.

EE is a purely quantum mechanical effect: may have complete information (0 entropy) of a given system but less (> 0 entropy) on a subsystem, which is not possible in classical mechanics.

• We are interested in spatial EE of (quantum mechanical) many-particle systems. There is large body of results, mostly numerical and conjectural and few rigorous results.

• We report on rigorous results (temperature $T = 0, T > 0$) for free (no pair interaction) Fermi gas in Euclidean space $\mathbb{R}^d, d \geq 1$ in thermal equilibrium, possibly subject to external electric or magnetic fields.

• Equilibrium states are characterized by two parameters (T, μ) (grand-canonical), or equivalently (T, ρ) (canonical), where $T \geq 0$ temperature, $\mu \in \mathbb{R}$ chemical potential, and $\rho > 0$ particle density. Write in all cases ω_T.

• Result for $T = 0$ was known since 2005, which we proved 2014. For $T > 0$ our result was new, also in physics literature.

• We also report on EE for ideal Fermi gas in $\mathbb{R}^2$ with constant magnetic field, which was only partially known in physics literature.

• Consider (bounded Borel) $\Lambda \subset \mathbb{R}^d$ and localize (or reduce, restrict) ω_T to Λ. Call this localized state $\omega_T \restriction \Lambda$.

• Local entropy, $\mathrm{S}(\omega_T, \Lambda)$, is defined as von Neumann entropy of localized state $\omega_T \restriction \Lambda$.

• Entanglement entropy (EE, or mutual information), $\mathrm{EE}(\omega_T, \Lambda)$ measures correlations of particles inside Λ with those outside Λ, that is, with $\Lambda^{\complement} = \mathbb{R}^d \setminus \Lambda$. Loosely speaking

$$\mathrm{EE}(\omega_T, \Lambda) = \mathrm{S}(\omega_T, \Lambda) + \text{“}\mathrm{S}(\omega_T, \Lambda^{\complement}) - \mathrm{S}(\omega_T, \mathbb{R}^d)\text{”}\,. \tag{1}$$

At $T = 0$, ω_T being pure (ground) state,

$$\mathrm{S}(\omega_T) = 0\,, \qquad \mathrm{S}(\omega_T, \Lambda) = \mathrm{S}(\omega_T, \Lambda^{\complement})\,. \tag{2}$$

Therefore,

$$\mathrm{EE}(\omega_T, \Lambda) = 2\,\mathrm{S}(\omega_T, \Lambda) \tag{3}$$

and local entropy $\mathrm{S}(\omega_T, \Lambda)$ is usually called entanglement entropy. Another measure of correlations is logarithmic negativity, which is well-understood at $T = 0$ but not at $T > 0$.

Goal is to understand these entropies for large Λ (and fixed ω_T). To this end, we fix Λ, introduce scaling parameter $L > 0$ and study asymptotic behavior of $\mathrm{S}(\omega_T, L\Lambda)$ and $\mathrm{EE}(\omega_T, L\Lambda)$ as $L \to \infty$.

Most studies deal with EE for ground states, also due to surprising logarithmic enhancement in some important cases as we'll see. Expect for $T > 0$

$$\mathrm{S}(\omega_T, L\Lambda) = s(T)|\Lambda|L^d + \eta(T, \partial\Lambda)L^{d-1} + o(L^{d-1}) \tag{4}$$

$$\mathrm{EE}(\omega_T, L\Lambda) = 2\eta(T, \partial\Lambda)L^{d-1} + o(L^{d-1})\,. \tag{5}$$

Here, $s(T) = \frac{\partial p(T)}{\partial T}$ is usual thermal entropy density, given as derivative of pressure p. Description of $\eta(T, \partial\Lambda)$ is rather complicated but simplifies as $T \downarrow 0$.

At $T = 0$ scaling of EE might be different. Aside from missing leading volume term ($s(0) = 0$), truely leading term may have extra $\ln(L)$-term, which depends on model.

As we deal with non-interacting fermions we need to specify one-particle Hamiltonian, H, only. Focus on

- Laplacian $\mathrm{H} = -\Delta$, or more generally $\mathrm{H} = K(-\mathrm{i}\nabla)$ with "dispersion relation" $K : \mathbb{R}^d \to \mathbb{R}$ on $\mathsf{L}^2(\mathbb{R}^d)$, f.i. $K(\xi) = \xi^2$.
- Landau Hamiltonian $\mathrm{H}_B = (-\mathrm{i}\nabla - a)^2$ on $\mathsf{L}^2(\mathbb{R}^2)$ with (constant) magnetic field B.

Introduce Fermi function $F_T : \mathbb{R} \to [0, 1]$,

$$f_T(E) := \begin{cases} (1 + \mathrm{e}^{E/T})^{-1} & \text{for } T > 0 \\ \Theta(-E) & \text{for } T = 0 \end{cases},$$

where Θ is Heaviside's unit step function. One-particle density operator

$$\mathrm{D}_T := f_T(\mathrm{H} - \mu\mathbb{1}) = (1 + \exp((\mathrm{H} - \mu\mathbb{1})/T))^{-1}$$

on $\mathsf{L}^2(\mathbb{R}^d)$ describes all correlations of equilibrium state ω_T. Here, $\mathbb{1}$ is identity on $\mathsf{L}^2(\mathbb{R}^d)$. Correlation functions:

$$\omega_T\big[a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_m)\big] = \delta_{n,m}\det\langle g_i, \mathrm{D}_T f_j\rangle \qquad (6)$$

with Fermi creation/annihilation operators $a^*(f)/a(g)$ and $f, g \in \mathsf{L}^2(\mathbb{R}^d)$. Fermi algebra

$$\mathcal{A}_{\mathcal{H}} = \mathsf{span}\big\{a^*(f), a(g) : f, g \in \mathcal{H} \subseteq \mathsf{L}^2(\mathbb{R}^d)\big\}\,.$$

D_T satisfies $0 \le \mathrm{D}_T \le \mathbb{1}$, $\mathrm{D}_0 = 1_{(-\infty,\mu]}(\mathrm{H})$ is (Fermi) projection.

Localization of states ω on $\mathcal{A}_{\mathcal{H}}$: For $\Lambda \subset \mathbb{R}^d$, let $\mathcal{H}_1 := \mathsf{L}^2(\Lambda)$, $\mathcal{H}_2 := \mathsf{L}^2(\Lambda^{\complement})$. Then $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^d) = \mathcal{H}_1 \oplus \mathcal{H}_2$ implies

$$\mathcal{A}_{\mathcal{H}} = \mathcal{A}_{\mathcal{H}_1} \otimes \mathcal{A}_{\mathcal{H}_2}\,.$$

Locally to Λ (resp. $\Lambda^{\complement}$) reduced many-fermion state $\omega \upharpoonright \Lambda$ is reduced (marginal, partial) state

$$(\omega \upharpoonright \Lambda)(A) := \omega(A \otimes \mathbb{1})\,, \quad A \in \mathcal{A}_{\mathcal{H}_1}\,,$$
$$(\omega \upharpoonright \Lambda^{\complement})(A') := \omega(\mathbb{1} \otimes A')\,, \quad A' \in \mathcal{A}_{\mathcal{H}_2}\,.$$

$\omega \restriction \Lambda$ and $\omega \restriction \Lambda^{\complement}$ are quasi-free (as in (6)), determined by one-particle density operators

$$\mathrm{D}(\Lambda) := \mathbb{1}_\Lambda \,\mathrm{D}\, \mathbb{1}_\Lambda \quad \mathrm{D}(\Lambda^{\complement}) := \mathbb{1}_{\Lambda^{\complement}} \,\mathrm{D}\, \mathbb{1}_{\Lambda^{\complement}}$$

Here, D is one-particle density operator of ω and $\mathbb{1}_\Lambda$ is multiplication operator with indicator function 1_Λ on $\mathsf{L}^2(\mathbb{R}^d)$:

$$(\omega \restriction \Lambda)\big[a^*(f_1)\cdots a^*(f_m)a(g_1)\cdots a(g_m)\big] = \delta_{n,m} \det\langle g_i, \mathrm{D}(\Lambda) f_j\rangle$$

for all $f_i, g_j \in \mathsf{L}^2(\Lambda)$. We define von Neumann entropy of any quasi-free state ω determined by D,

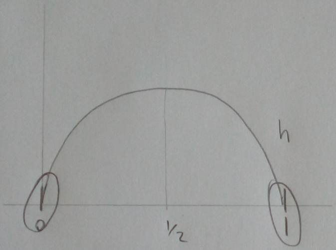
$$\mathrm{S}(\omega) := \mathrm{S}(\mathrm{D}) := \mathrm{tr}\, h(\mathrm{D}) \;\in\; [0,\infty]\,,$$

where $h : [0,1] \to [0, \ln(2)]$ is binary entropy function defined by $h(0) := 0$, $h(1) := 0$ and

$$h(t) := -t\ln(t) - (1-t)\ln(1-t)\,, \qquad t \in\,]0,1[\,.$$

Von-Neumann entropy function

$$h(t) = -t\ln(t) - (1-t)\ln(1-t)$$



smooth except at 0 and 1,

$h(t) \leq C_\alpha t^\alpha (1-t)^\alpha$ for any $\alpha < 1$.

Define local entropy of (quasi-free) state ω resp. D reduced to Λ as

$$\mathrm{S}(\omega \restriction \Lambda) := \mathrm{S}(\mathrm{D}(\Lambda)) := \operatorname{tr} h(\mathrm{D}(\Lambda))$$

To define EE we cannot use (1) directly, because two of the entropy terms are typically unbounded. So first we introduce for $\Omega = \Lambda$ or $\Omega = \Lambda^{\complement}$ entropic difference operator

$$\Delta(\omega, \Omega) := h\big[\mathbb{1}_\Omega \, \mathrm{D} \, \mathbb{1}_\Omega\big] - \mathbb{1}_\Omega \, h(\mathrm{D}) \, \mathbb{1}_\Omega \, ,$$

and then define

$$\mathrm{EE}(\omega, \Lambda) := \operatorname{tr} \Delta(\omega, \Lambda) + \operatorname{tr} \Delta(\omega, \Lambda^{\complement}) \, .$$

By operator-concavity of h, $\Delta(T, \Omega) \geq 0$ and hence $\mathrm{EE}(\omega, \Lambda) \geq 0$.

For $\omega = \omega_T$ equilibrium state at temperature $T > 0$ we prove that $\Delta(\omega_T, \Omega)$ is trace-class and a-priori estimate

$$\operatorname{tr} \Delta(\omega_T, L\Omega) \leq C L^{d-1}$$

if $\Omega = \Lambda$ or $\Lambda^{\complement}$ is bounded set in $\mathbb{R}^d$. This implies an area-law bound for EE in line with (5).

Reason for this definition of EE:

$$\begin{aligned}
\operatorname{tr} \Delta(\omega, \Lambda) + \operatorname{tr} \Delta(\omega, \Lambda^{\complement}) &= \operatorname{tr} \left\{ h[\mathbb{1}_\Lambda \, \mathrm{D} \, \mathbb{1}_\Lambda] - \mathbb{1}_\Lambda \, h(\mathrm{D}) \, \mathbb{1}_\Lambda \right\} \\
&\quad + \operatorname{tr} \left\{ h[\mathbb{1}_{\Lambda^{\complement}} \, \mathrm{D} \, \mathbb{1}_{\Lambda^{\complement}}] - \mathbb{1}_{\Lambda^{\complement}} \, h(\mathrm{D}) \, \mathbb{1}_{\Lambda^{\complement}} \right\} \\
&= \mathrm{S}(\omega \restriction \Lambda) + \text{``}\mathrm{S}(\omega \restriction \Lambda^{\complement}) - \mathrm{S}(\omega)\text{''} \, .
\end{aligned}$$

Individual entropies are typically infinite but terms are reorganized in Δ-operators such that they become trace-class.

To determine its precise scaling we need formula by Widom and some more definitions. For $\{r,s\} \subset [0,1]$ and f.i. h entropy function, define

$$U(r,s;h) := \tfrac{1}{4\pi^2} \int_0^1 \mathrm{d}t \, \frac{h((1-t)r+ts) - (1-t)h(r) - th(s)}{t(1-t)} \, .$$

By concavity of $h, U(r,s;h) \geq 0$. For function $g : \mathbb{R} \to [0,1]$, let

$$\mathcal{U}[g] := \mathcal{U}[g;h] := \int_{\mathbb{R}^2} \mathrm{d}u\mathrm{d}v \, \frac{U(g(u),g(v);h)}{(u-v)^2} \;\geq\; 0$$

as principle value. For $d = 1$ and $\Lambda \subset \mathbb{R}$, set

$$\eta(T,\partial\Lambda) := \mathcal{U}[f_T \circ (K-\mu)] \, |\partial\Lambda| \geq 0 \, .$$

If $d \geq 2, x \in \partial\Lambda$ and $\xi \in \mathsf{T}_x^*(\partial\Lambda) \cong \mathbb{R}^{d-1}$ then define firstly reduced one-dimensional symbol $f_{T;(x,\xi)} : \mathbb{R} \to \mathbb{R}$

$$\nu \mapsto f_{T;(x,\xi)}(\nu) := f_T(K(\xi + \nu \cdot n_x) - \mu)\,.$$

So, if $K(\xi) = \xi^2$ then $K(\xi + \nu \cdot n_x) = \xi^2 + \nu^2$ and

$$f_{T;(x,\xi)}(\nu) = f_T(\xi^2 + \nu^2 - \mu)\,.$$

Secondly, let

$$\begin{aligned} \eta(T, \partial\Lambda) &:= (2\pi)^{1-d} \int_{\partial\Lambda} \mathrm{d}\sigma(x) \int_{\mathbb{R}^{d-1}} \mathrm{d}\xi\, \mathcal{U}[f_{T;(x,\xi)}] \\ &= \int_{\mathsf{T}^*(\partial\Lambda)} \mathsf{d}X\, \mathcal{U}[f_{T;X}]\,. \end{aligned}$$

Non-trivial to prove that $\eta(T, \partial\Lambda) < \infty$.

Theorem ($d = 1$, 2016; Sobolev for $d > 1$, 2017)

Let $\Lambda \subset \mathbb{R}^d$ be bounded with piece-wise C^1-boundary and finitely many connected components. Let K be smooth and polynomially bounded, $\mathrm{D}_T = (1 + \exp((K(-\mathrm{i}\nabla) - \mu\mathbb{1})/T))^{-1}$ corresponding to equilibrium state ω_T and $\Omega = \Lambda$ or its complement $\Lambda^{\complement}$. Then, as $L \to \infty$

$$\begin{aligned}\operatorname{tr}\Delta(\omega_T, L\Omega) &= \operatorname{tr}\left\{h\big[\mathbb{1}_{L\Omega}\,\mathrm{D}_T\,\mathbb{1}_{L\Omega}\big] - \mathbb{1}_{L\Omega}\,h(\mathrm{D}_T)\,\mathbb{1}_{L\Omega}\right\}\\ &= \eta(T, \partial\Lambda)L^{d-1} + o(L^{d-1})\,.\end{aligned}$$

Area-law scaling of EE in equilibrium state at $T > 0$, as $L \to \infty$

$$\begin{aligned}\mathrm{EE}(\omega_T, L\Lambda) &= \operatorname{tr}\Delta(\omega_T, L\Lambda) + \operatorname{tr}\Delta(\omega_T, L\Lambda^{\complement})\\ &= 2\,\eta(T, \partial\Lambda)L^{d-1} + o(L^{d-1})\end{aligned}$$

and two-term asymptotic expansion of local entropy of equilibrium state ω_T at chemical potential μ:

$$\begin{aligned}
\mathrm{S}(\omega_T, L\Lambda) &= \mathrm{tr}\left[h(\mathbb{1}_{L\Lambda}\, f_T(K(-\mathrm{i}\nabla) - \mu\mathbb{1})\, \mathbb{1}_{L\Lambda})\right] \\
&= \mathrm{tr}\,\Delta(\omega_T, L\Lambda) + \mathrm{tr}\left[\mathbb{1}_{L\Lambda}\,(h \circ f_T)(K(-\mathrm{i}\nabla) - \mu\mathbb{1})\, \mathbb{1}_{L\Lambda}\right] \\
&= s(T)|\Lambda| L^d + \eta(T, \partial\Lambda) L^{d-1} + o(L^{d-1})\,,
\end{aligned}$$

with

$$s(T) := (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{d}\xi\,(h \circ f_T)(K(\xi) - \mu)$$

usual thermal entropy density.

Leading term, long known in physics, was proved in 1993/1998, next-to-leading $\eta(T, \partial\Lambda)$ term is new.

Theorem ($T = 0$, 2014)

Let $\Gamma \subset \mathbb{R}^d$ *be bounded with piece-wise* C^3*-boundary* $\partial\Gamma$, $\mathrm{D}_0 = \mathbb{1}_\Gamma(-\mathrm{i}\nabla)$ *one-particle density operator of ground state, and let* $\Lambda \subset \mathbb{R}^d$ *be bounded with piece-wise* C^1*-boundary* $\partial\Lambda$*. Then, local entropy behaves asymptotically as* $L \to \infty$

$$
\begin{aligned}
\mathrm{S}(\omega_0, L\Lambda) &= \mathrm{S}(\mathrm{D}_0(L\Lambda)) = \operatorname{tr} h\big[\mathbb{1}_{L\Lambda}\, \mathbb{1}_\Gamma(-\mathrm{i}\nabla)\, \mathbb{1}_{L\Lambda}\big] \\
&= h(1)\,|\Lambda|\,|\Gamma/(2\pi)|\, L^d \\
&\quad + \underbrace{U(0,1;h)}_{=\frac{1}{12}}\, \underbrace{(2\pi)^{1-d} \int_{\partial\Gamma\times\partial\Lambda} \mathrm{d}\tau(\xi)\mathrm{d}\sigma(x)\, |n_\xi \cdot n_x|}_{=:\mathcal{I}(\partial\Gamma,\partial\Lambda)}\, L^{d-1} \ln(L) \\
&\quad + o(L^{d-1} \ln(L))\,.
\end{aligned}
$$

n_ξ and n_x are unit normal vectors at $\xi \in \partial\Gamma$ resp. $x \in \partial\Lambda$, and τ and σ are surface measures on $\partial\Gamma$ resp. $\partial\Lambda$.

- Since $h(1) = 0$, "leading" (volume) Weyl-term vanishes and leading term of entropy is $O(L^{d-1}\ln(L))$; extra $\ln(L)$ is due to step discontinuity of symbol 1_Γ as function of momentum ξ.
- "Fermi-sea" w.r.t. dispersion K: $\Gamma = \{\xi \in \mathbb{R}^d : K(\xi) \leq \mu\}$.
- For spherical Fermi "surface" $\partial\Gamma = p_\mathsf{F}\mathbb{S}^{d-1}$ with radius $p_\mathsf{F} > 0$,

$$\mathcal{I}(\partial\Gamma, \partial\Lambda) = [2^{2-d}/((d-1)/2)!]\,(p_\mathsf{F}/\pi)^{(d-1)/2}\,|\partial\Lambda|$$

  implying logarithmically enhanced area law.
- For $d = 1$ and with $|\partial\Lambda|$ (even) number of boundary points, and

$$\mathcal{I}(\partial\Gamma, \partial\Lambda) = |\partial\Gamma|\,|\partial\Lambda|\,.$$

Further remarks:

- In 2005, Gioev–Klich found above connection between EE and conjectured formula by Widom from 1982. Latter concerns two-term Szegö asymptotics for multi-dimensional Wiener–Hopf (Toeplitz) operators. This sparked my interest.
- Our joint proof uses Sobolev's remarkable proof of remarkable conjecture by Widom for smooth functions "h".
- Also based upon classical bounds by Birman–Solomyak on singular values of compact integral operators to deal with non-smooth h.
- Recent extension by P. Müller and R. Schulte to one-particle Hamiltonian $\mathrm{H} = -\Delta + V$ on $\mathsf{L}^2(\mathbb{R}^d)$ with compactly supported V and Fermi-projection $\mathrm{D}_0 = 1_{(-\infty,\mu]}(\mathrm{H})$.

Small temperatures

L^{d-1}-scaling of entropy at $T > 0$ versus $L^{d-1}\ln(L)$-scaling at $T = 0$. How to reconcile this?

As $T \downarrow 0$, leading asymptotic coefficient $s(T)$ of volume term L^d goes to zero but coefficient $\eta(T, \partial\Lambda)$ of surface term of order L^{d-1} displays logarithmic singularity in T.

With Fermi-sea $\Gamma := \{\xi \in \mathbb{R}^d : K(\xi) \le \mu\}$,

$$\eta(T, \partial\Lambda) \quad = \quad U(0,1;h)\,\mathcal{I}(\partial\Gamma, \partial\Lambda)\,\ln(1/T) + O_T(1)\,.$$

Recall, at $T = 0$,

$$\mathrm{S}(\mathrm{D}_0(L\Lambda)) = U(0,1;h)\,\mathcal{I}(\partial\Gamma, \partial\Lambda)\,L^{d-1}\ln(L) + o(L^{d-1}\ln(L))\,.$$

So if we identify $1/T = L$ we recover logarithmically enhanced area-law scaling at zero temperature.

Landau Hamiltonian

Consider free (pairwise non-interacting), spinless, charged fermions confined to Euclidean plane $\mathbb{R}^2$ subject to perpendicular constant magnetic field of strength $B > 0$. Single particle Hamiltonian,

$$\mathsf{H} := (-\mathrm{i}\nabla - a)^2$$

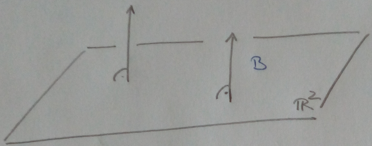
on $\mathsf{L}^2(\mathbb{R}^2)$. For $x = (x_1, x_2) \in \mathbb{R}^2$, $\nabla := (\partial/\partial x_1, \partial/\partial x_2)$, gauge $a(x) = (a_1(x), a_2(x)) := B/2(-x_2, x_1)$. Spectral decomposition (Fock, Landau)

$$\mathsf{H} = B\sum_{\ell=0}^{\infty}(2\ell+1)\mathsf{P}_\ell$$

with eigenvalues $\{B, 3B, 5B, \ldots\}$ and (infinite-dimensional) spectral projections P_ℓ, with integral kernel

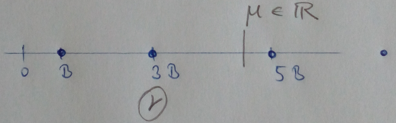
$$\mathsf{P}_\ell(x,y) = \frac{B}{2\pi}\mathcal{L}_\ell(B\|x-y\|^2/2)\,\mathrm{e}^{-\frac{B}{4}\|x-y\|^2 + \mathrm{i}\frac{B}{2}(x_1y_2 - x_2y_1)}\,.$$

$$H = (-i\nabla - a)^2$$



$$\frac{\partial}{\partial x_1} a_2(x) - \frac{\partial}{\partial x_2} a_1(x) = B.$$

Spectrum of H



Fermi projection: $\mathbb{1}_{(-\infty,\mu]}(H) = P_{\leq \gamma}$

$\mathcal{L}_\ell(t) := \sum_{j=0}^{\ell} \frac{(-1)^j}{j!}\binom{\ell}{\ell-j} t^j$ Laguerre polynomial, $t \in [0,\infty)$.

Ground state of free fermions at some Fermi energy $\mu \geq B$ is characterized by Fermi projection on $\mathsf{L}^2(\mathbb{R}^2)$,

$$1_{(-\infty,\mu]}(\mathsf{H}) = \sum_{\ell=0}^{\nu} \mathsf{P}_\ell =: \mathsf{P}_{\leq\nu}$$

with $\nu := \lfloor (\mu/B - 1)/2 \rfloor$ integer part of $(\mu/B-1)/2 \geq 0$.

Let $\Lambda \subseteq \mathbb{R}^d$ Borel set. Denote by $\mathbb{1}_\Lambda$ multiplication operator on $\mathsf{L}^2(\mathbb{R}^d)$ by indicator function 1_Λ on $\mathbb{R}^d$. Ground state of fermions localized to $\Lambda \subseteq \mathbb{R}^2$ is characterized by local(ized) Fermi projection

$$0 \leq \mathbb{1}_\Lambda\, 1_{(-\infty,\mu]}(\mathsf{H})\, \mathbb{1}_\Lambda = \mathbb{1}_\Lambda \mathsf{P}_{\leq\nu} \mathbb{1}_\Lambda =: \mathsf{P}_{\leq\nu}(\Lambda) \leq \mathbb{1}_{\mathbb{R}^2}\,.$$

Recall local von-Neumann entropy (or entanglement entropy)

$$\mathrm{S}_\nu(\Lambda) := \operatorname{tr} h(\mathsf{P}_{\leq\nu}(\Lambda))\,,$$

with entropy function $h(t) = -t\ln(t) - (1-t)\ln(1-t)$ on $[0,1]$. Clearly, $\mathrm{S}_\nu(\mathbb{R}^2) = 0$, and, in general, $\mathrm{S}_\nu(\Lambda) \in [0,\infty]$. We prove

- $\mathrm{S}_\nu(\Lambda) < \infty$ if Λ is bounded (often ignored but non-trivial!);
- leading asymptotic "area-law" scaling as $L \to \infty$

$$\mathrm{S}_\nu(L\Lambda) = L\sqrt{B}\,|\partial\Lambda|\,\mathsf{M}_\nu + o(L)\,.$$

- Coefficient M_ν independent of L and Λ.
- We assume boundary curve $\partial\Lambda$ to be C^3-smooth. Later, we identify $\mathsf{M}_\nu = \mathsf{M}_\nu(h)$ of certain functional $f \mapsto \mathsf{M}_\nu(f)$.

In this talk we consider only $\ell = 0$, $\mathcal{L}_0 = 1$. M_ν for $\nu > 0$ is more complicated due to mixing of Landau level projections.

Functional $f \mapsto \mathsf{M}_0(f)$ is of form

$$\mathsf{M}_0(f) = \int_{\mathbb{R}} \frac{\mathrm{d}\xi}{2\pi} \left[f(\lambda_0(\xi)) - f(1)\lambda_0(\xi) \right],$$

with

$$\lambda_0(\xi) = \pi^{-1/2} \int_{\xi}^{\infty} \mathrm{d}t \exp(-t^2) \,.$$

Here, $f : [0,1] \to \mathbb{C}$ continuous on closed unit interval $[0,1]$, right-sided differentiable at 0, left-sided differentiable at 1, and satisfies $f(0) = 0$. Then, $|\mathsf{M}_0(f)| < \infty$.

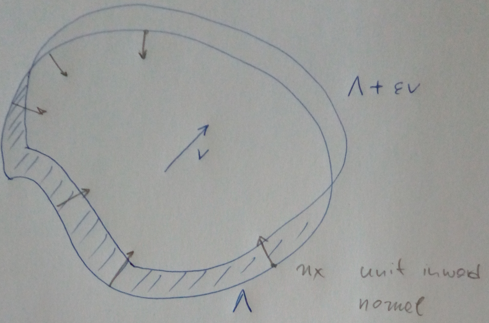
We assume furthermore that $\Lambda \subset \mathbb{R}^2$ is bounded C^3-region: union of finitely many connected open sets with disjoint closures. Boundary curve $\partial\Lambda$ is C^3.

Theorem ("Smooth f", 2020)

Under the above assumptions on Λ and f we have 2-term asymptotic expansions as $L \to \infty$,

$$\mathrm{tr} f(\mathrm{P}_0(L\Lambda)) = L^2 B \frac{|\Lambda|}{2\pi} f(1) + L\sqrt{B} |\partial\Lambda| \, \mathsf{M}_0(f) + o(L) \, .$$

Proof ist first done for polynomial functions f, where $o(L)$ term is, in fact, $o(1)$. Extension of asymptotic expansion from polynomials to functions f in theorem uses Weierstraß approximation: write $f(t) - tf(1) =: b(t)t(1-t)$ and approximate b by polynomial p uniformly on $[0,1]$.



$$|\Lambda \setminus \Lambda_\varepsilon| = \varepsilon^{d-1} \int_{\partial\Lambda} \max(0, v \cdot n)\, dA(x)$$

$$+ o(\varepsilon^{d-1}).$$

$d = 2, \quad \varepsilon = 1/\sqrt{L^2 B}.$

For polynomials $f(t) = t^{r+1}$, $r \in \mathbb{N}$, we follow approach of Roccaforte for Wiener–Hopf/Toeplitz operators. For given $v_1, \dots, v_r$ in $\mathbb{R}^2$, let $\Lambda_\varepsilon := \Lambda \cap (\Lambda + \varepsilon v_1) \cap \cdots \cap (\Lambda + \varepsilon v_r)$ intersection of Λ with its r translates; $\varepsilon = 1/(L\sqrt{B})$. Then,

$$|\Lambda \setminus \Lambda_\varepsilon| = \varepsilon \int_{\partial\Lambda} \mathrm{d}A(x) \, \max\left\{0, \langle v_1 | n_x \rangle, \dots, \langle v_r | n_x \rangle\right\} + o(\varepsilon)\,.$$

n_x is inward unit normal vector at $x \in \partial\Lambda$.

Alas, entropy function h is not (one-sided) differentiable at 0 and 1 but nevertheless, we have

Theorem (Entropy: main result, 2020)

Let $\Lambda \subset \mathbb{R}^2$ be bounded C^3*-region. Then, as $L \to \infty$*

$$\mathrm{S}_0(L\Lambda) = \operatorname{tr} h(\mathrm{P}_0(L\Lambda)) = L\sqrt{B}\,|\partial\Lambda|\,\mathsf{M}_0(h) + o(L)\,. \qquad (7)$$

In fact, $\mathsf{M}_0(h) = 0{,}203\ldots$

"Leading" volume/area term is zero because $h(1) = 0$.

To deal with this asymptotics write

$$h = (1 - \zeta_\varepsilon)h + \zeta_\varepsilon h$$

with smooth function ζ_ε, $0 \le \zeta_\varepsilon(t) \le 1$ supported on trouble-zone $[0, \varepsilon] \cup [1 - \varepsilon, 1]$.

- Apply first theorem with smooth $f = (1 - \zeta_\varepsilon)h$.
- For second term use

$$0 \le h(t) \le C\, t^{1/3}(1 - t)^{1/3}\,.$$

This shows $|(\zeta_\varepsilon h)(t)| \le C\varepsilon^{1/3} t^{1/3}(1 - t)^{1/3}$ and hence

$$\begin{aligned}\big\|(\zeta_\varepsilon h)(\mathsf{P}_0(L\Lambda))\big\|_1 &\le C\varepsilon^{1/3}\big\|\mathsf{P}_0(L\Lambda)\big(\mathbb{1} - \mathsf{P}_0(L\Lambda)\big)\big\|_{1/3}^{1/3} \\ &= C\varepsilon^{1/3}\big\|\mathbb{1}_{L\Lambda}\mathsf{P}_0\big(\mathbb{1} - \mathbb{1}_{L\Lambda}\big)\big\|_{2/3}^{2/3}\,.\end{aligned}$$

For $0 < p < \infty$, compact operator T with singular values $s_n(\mathrm{T})$, Schatten–von Neumann (quasi-)norm

$$\|\mathrm{T}\|_p := \left[\sum_{n=1}^{\infty} s_n(\mathrm{T})^p\right]^{\frac{1}{p}} < \infty\,.$$

If $p \geq 1$, then $\|\cdot\|_p$ is norm. If $0 < p < 1$, it is quasi-norm which satisfies p-triangle inequality

$$\|\mathrm{T}_1 + \mathrm{T}_2\|_p^p \leq \|\mathrm{T}_1\|_p^p + \|\mathrm{T}_2\|_p^p\,.$$

Theorem

Let $\Lambda \subset \mathbb{R}^2$ be bounded Lipschitz-region, $\ell \in \mathbb{N}_0$, $p \in (0,1]$ and $L_0 > 0$. Then there exists constant C, depending on Λ, ℓ, and L_0, such that for any $L \geq L_0$,

$$\|\mathbb{1}_{L\Lambda}\mathrm{P}_\ell(\mathbb{1} - \mathbb{1}_{L\Lambda})\|_p^p \leq CL\,.$$

- Bound is due to fast (exponential) decay of off-diagonal part of kernel of Landau-projections. Not true if $B = 0$, where extra $\ln(L)$ appears.
- This bounds remainder term $\left\|(\zeta_\varepsilon h)(\mathsf{P}_0(L\Lambda))\right\|_1 \leq C\varepsilon^{1/3}L$, which vanishes as $\varepsilon \downarrow 0$.

Previous results on local entropy in magnetic field:

- I.D. Rodriguez and G. Sierra in 2009 found formula for local entropy for special domains Λ and lowest Landau level $\ell = 0$.
- L. Charles and B. Estienne proved in 2018 this formula by completely different methods (for $\ell = 0$).

$\mathsf{D}(0,r)$ disk of radius r with center 0.

Lemma (Birman–Solomyak)

Let $\mathrm{Z} : \mathrm{L}^2(\mathbb{R}^2) \to \mathrm{L}^2(\mathsf{D}(0,r))$ be integral operator defined by complex-valued kernel $\mathrm{Z}(x,y)$ obeying for some $\gamma \in \mathbb{N}_0$

$$N_\gamma(\mathrm{Z}) := \Big[\sum_{0 \le s,t \le \gamma} \int_{\mathbb{R}^2} \mathrm{d}y \int_{\mathsf{D}(0,r)} \mathrm{d}x \left| \frac{\partial^s}{\partial x_1^s} \frac{\partial^t}{\partial x_2^t} \mathrm{Z}(x,y) \right|^2 \Big]^{\frac{1}{2}} < \infty \,.$$

Then singular values $s_n(\mathrm{Z})$ of Z satisfy bound

$$s_n(\mathrm{Z}) \le C n^{-\frac{1+\gamma}{2}} N_\gamma(\mathrm{Z}) \,, \quad n \in \mathbb{N} \,,$$

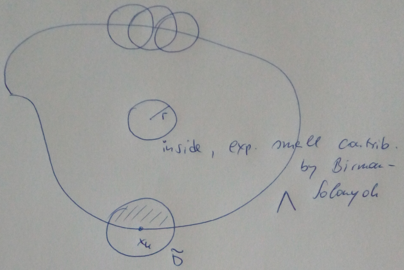
with constant C dependent on r but independent of integral kernel.

Then, for $p > 0$,

$$\|\mathrm{Z}\|_p^p \le C N_\gamma(\mathrm{Z}) \sum_{n \ge 1} n^{-\frac{1+\gamma}{2} p} < \infty$$

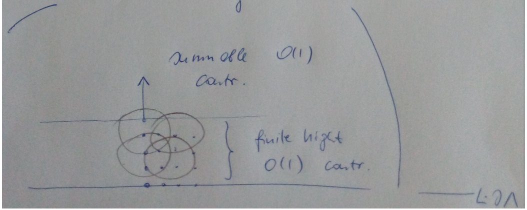
if $\gamma > (2/p) - 1$.

$$\Lambda \subset \bigcup_{i\in I,\ \text{finite}} D_i$$



$$\Lambda \cap \tilde{D} = \{x = (x', x'') : x'' > \Phi(x')\} \cap \tilde{D}$$

$$L(\Lambda \cap \tilde{D}) \subseteq \bigcup_j D(x_j, 1)$$



For $R > r > 0$ we immediately obtain for any $0 < p \le 1$

$$\|\mathbb{1}_{\mathsf{D}(0,r)} \mathrm{P}_\ell (\mathbb{1} - \mathbb{1}_{\mathsf{D}(0,R)})\|_p \le C_{p,r} \exp(-C(R-r)^2)\,,$$

$C_{p,r}$ independent of R. Cover Λ by finitely many disks $\mathsf{D}(x_k, r)$:

1. $x_k \in \partial\Lambda$ and inside each disk $\widetilde{\mathsf{D}} = \mathsf{D}(x_k, r)$, with appropriate choice of coordinates,

   $$\Lambda \cap \widetilde{\mathsf{D}} = \{x = (x', x'') \in \mathbb{R}^2 : x'' > \Phi(x')\} \cap \widetilde{\mathsf{D}}$$

   with Lipschitz function Φ, or
2. $\mathsf{D}(x_k, r) \subset \Lambda$ and $\operatorname{dist}\big(\mathsf{D}(x_k, r), \mathbb{R}^2 \setminus \Lambda\big) > 0$.

Cover the scaled disks $L\mathsf{D}(x_k, r)$ by disks of radius 1. Disks inside $L\Lambda$ will have exponentially small contributions,

$$\|\mathbb{1}_{\mathsf{D}(0,r)} \mathrm{P}_\ell (\mathbb{1} - \mathbb{1}_{L\Lambda})\|_p \le C \exp(-CL^2)\,.$$

Each disk at boundary at boundary $\partial(L\Lambda)$ contributes at order 1. Number of such disks is of order of length of boundary, $L|\partial\Lambda|$.

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