Incompatibility of Frequency Splitting and Spatial Localization: A quantitative analysis of Hegerfeldt's theorem

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funded by the SNSF grant P2SKP2_178198

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November 20, 2020

My Goal

In my talk

- I will give you a rough idea of the problem of localization in Quantum Mechanics.
- I will introduce a novel type of PDE estimates for wave equations in Minkowski space.

Outline

1. Introduction
   - Localization in Quantum Mechanics
   - Hegerfeldt's Theorem
2. Frequency Splitting
   - Main Theorem
3. Outlook

Localization in Quantum Mechanics

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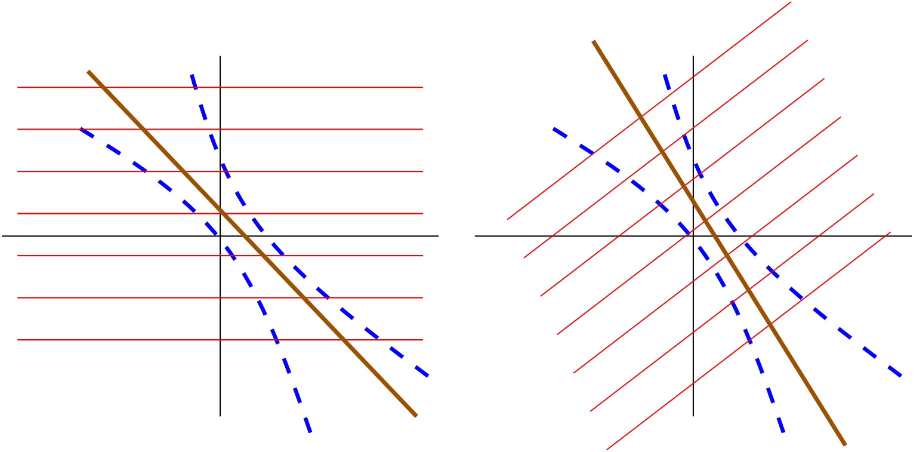
Definition: *A particle state is said to be localized in V at time t if the probability of finding the particle in V is 1. It is said to be not in V at time t if the probability is 0.*

Localization in the Relativistic Setting



Hegerfeldt's Theorem

Causality Condition If at time $t_0 = 0$ a particle state is localized in V, then there is a constant $r = r_t$, such that, at time $t > 0$, the particle, when translated by $\vec{a}$, $|\vec{a}| \geq r_t$ is not in V.

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Theorem (Hegerfeldt 1974) *In a relativistic theory there is no one-particle state localized in the finite space region V satisfying the causality condition.*

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Hegerfeldt implies that solutions of hyperbolic partial differential equations in $d + 1$-dimensional Minkowski space which have spatially compact support cannot be composed purely of positive (or similarly negative) frequencies.

Set Up

Let $B_1 = (-1, 1)$. We consider the Cauchy problem for the scalar wave equation with smooth, compactly supported initial data in B_1,

$$\begin{cases} (\partial_t^2 - \partial_x^2)\phi(t, \vec{x}) = 0, \\ \phi|_{t=0} = \phi_0 \in C_0^\infty(B_1)\,, \\ \partial_t \phi|_{t=0} = \phi_1 \in C_0^\infty(B_1)\,. \end{cases} \tag{1}$$

We denote the energy of the solution by

$$E(\phi) := \frac{1}{2} \int_{B_1} \big((\partial_t \phi)^2 + (\partial_x \phi)^2\big)(t, x)\, dx. \tag{2}$$

Set Up

Taking the spatial spatial Fourier transform

$$\hat{\phi}(t,k) = \int_{B_1} \phi(t,x)\, e^{-ikx}\, dx\,, \tag{3}$$

we can split

$$\hat{\phi}(t,k) = \hat{\phi}_+(t,k) + \hat{\phi}_-(t,k) \tag{4}$$

with

$$\hat{\phi}_\pm(t,k) := \frac{1}{2}\, e^{\mp i\omega t}\Big(\hat{\phi}_0(k) \pm \frac{i}{\omega}\, \hat{\phi}_1(k)\Big)\,, \tag{5}$$

where $\omega \geq 0$ denotes the absolute value of the frequency, i.e.

$$\omega = \omega(k) := |k|\,. \tag{6}$$

Set Up

By Plancherel's theorem

$$
\begin{aligned}
E(\phi) &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\omega^2 \left|\hat{\phi}_0(k)\right|^2 + \left|\hat{\phi}_1(k)\right|^2\right) && (7)\\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \omega^2 \left(\left|\hat{\phi}_+(t,k)\right|^2 + \left|\hat{\phi}_-(t,k)\right|^2\right). && (8)
\end{aligned}
$$

Therefore

$$
E(\phi) = E(\phi_+) + E(\phi_-) \qquad \text{with} \qquad E(\phi_\pm) := \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \omega^2 \left|\hat{\phi}_\pm(t,k)\right|^2. \tag{9}
$$

Apriory Bounds

$$\phi(x) := (K_0 f)(x) = \int_M K_0(x,y)\, f(y)\, d^{d+1}y \tag{10}$$

Taking the Fourier transform, the convolution in (10) becomes a multiplication in momentum space, i.e.

$$\hat{\phi}(p) = \hat{K}_0(p)\, \hat{f}(p)\,, \text{ where } \hat{K}_0(p) = c\, \delta\big(\langle p, p\rangle\big)\ \big(\Theta(p^0) - \Theta(-p^0)\big) \tag{11}$$

We choose the following family of source functions

$$f_\zeta(x) = g(x)\, \exp\big(-i\zeta\,(x^0 + x^1)\big) \Rightarrow \hat{f}_\zeta(p) = \hat{g}\big(p^0 - \zeta, p^1 + \zeta, p^2, \ldots, p^d\big) \tag{12}$$

with a fixed test function g and a real parameter ζ.

$$\lim_{\zeta\to\infty} \frac{E(\phi_{-,\zeta})}{E(\phi_{+,\zeta})} = 0 \qquad \lim_{\zeta\to-\infty} \frac{E(\phi_{-,\zeta})}{E(\phi_{+,\zeta})} = \infty \tag{13}$$

Apriory Bounds

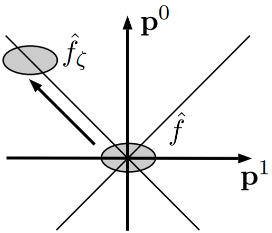

FIGURE 1. Shifting $\hat{f}_\zeta$ in momentum space. The shaded region indicates the neighborhood around the maximum of $\hat{f}_\zeta$, outside of which $\hat{f}_\zeta$ decays rapidly.

Main Theorem

Theorem (F. Finster – C.F.P.) *Assume that at a time t_0 a wave $\phi(t, x)$ is supported inside a ball of radius 1 further assume, the inequality*

$$E(\phi_-) \leq \varepsilon \, E(\phi_+)$$

holds for given $\varepsilon > 0$. This implies an a-priori estimate of the frequency distribution of ϕ of the form

$$\left\| \hat{\phi}(\omega, .) \right\|_* \leq g(\varepsilon, \omega) \sqrt{E(\phi)} \, .$$

Constant Bound

Define

$$\hat{h}(k) := \omega\,\hat{\phi}_{\pm}(k) \tag{14}$$

Then we get the first bound.

Lemma *For all* $\omega \in \mathbb{R}^+$,

$$|\hat{h}(\omega)| \leq \sqrt{2E(\phi)}. \tag{15}$$

Proof

According to the definition of $\hat{\phi}_\pm$

$$\left|\hat{h}_\pm(k)\right| = |k\,\hat{\phi}_\pm(k)| \le \frac{1}{2}\left(|k\,\hat{\phi}_0(k)| + |\hat{\phi}_1(k)|\right) \le \frac{1}{\sqrt{2}}\left(|k\,\hat{\phi}_0(k)|^2 + |\hat{\phi}_1(k)|^2\right)^{\frac{1}{2}}$$

The obtained Fourier transforms can be estimated pointwise by

$$\begin{aligned}
\left|k\,\hat{\phi}_0(k)\right| &\le \left|\int_{B_1} \partial_x\phi_0(x)\, e^{-ikx}\, dx\right| \le \int_{B_1} \left|\partial_x\phi_0(x)\right| dx \le \sqrt{2}\,\|\partial_x\phi_0\|_{L^2(B_1)} \\
\left|\hat{\phi}_1(k)\right| &\le \left|\int_{B_1} \phi_1(x)\, e^{-ikx}\, dx\right| \le \int_{B_1} \left|\phi_1(x)\right| dx \le \sqrt{2}\,\|\phi_1\|_{L^2(B_1)}\,.
\end{aligned}$$

Comparing with the definition of the energy evaluated at time $t = 0$ gives the result.

Taylor Expansion

For $1+1d$ we introduce the parity decomposition

$$\phi(t,x) = \phi^{even}(t,x) + \phi^{odd}(t,x)\,,$$

and get again a split of the energies

$$E(\phi_\pm) = E(\phi_\pm^{even}) + E(\phi_\pm^{odd}).$$

Since the initial data is compactly supported, its Fourier transform is real analytic and we can thus perform a Taylor expansion around $\omega = 0$

$$h^\bullet(\omega) := \sum_{n=0}^{\infty} a_n^\bullet \omega^n \ : \ \mathbb{R}^+ \to \mathbb{C} \tag{16}$$

A First Bound on Coefficients

Proposition *The coefficients in the power series for $h^\bullet(\omega)$ are bounded by*

$$|a_n^\bullet| \leq \frac{\sqrt{E(\phi^\bullet)}}{n!} .$$

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Proof Differentiating we get

$$\left|\hat{\phi}^{(n)}(k)\right| \leq \left| \int_{B_1} (-ix)^n \, \phi(x) \, e^{-ikx} \, dx \right| \leq \int_{B_1} \left|\phi(x)\right| dx \leq \sqrt{2} \, \|\phi\|_{L^2(B_1)} .$$

In particular, setting $k = 0$ we obtain

$$\left|a_n\right| \, n! = \left|\hat{\phi}^{(n)}(0)\right| \leq \sqrt{2} \, \|\phi\|_{L^2(B_1)} ,$$

Further

$$ik \, \hat{\phi}(k) = \sum_{n=1}^{\infty} d_n \, k^n \qquad \text{with} \qquad |d_n| \leq \frac{\sqrt{2}}{n!} \, \|\partial_x \phi\|_{L^2(B_1)} .$$

Proof

$$\left|a_{2\ell}^{\text{even}}\right| \leq \frac{1}{\sqrt{2}}\,\frac{\left\|\phi_1^{\text{even}}\right\|_{L^2(B_1)}}{(2\ell)!}\ , \qquad \left|a_{2\ell+1}^{\text{even}}\right| \leq \frac{1}{\sqrt{2}}\,\frac{\left\|\partial_x\phi_0^{\text{even}}\right\|_{L^2(B_1)}}{(2\ell+1)!}$$

$$\left|b_{2\ell+2}^{\text{odd}}\right| \leq \frac{1}{\sqrt{2}}\,\frac{\left\|\partial_x\phi_0^{\text{odd}}\right\|_{L^2(B_1)}}{(2\ell+2)!}\ , \qquad \left|b_{2\ell+1}^{\text{odd}}\right| \leq \frac{1}{\sqrt{2}}\,\frac{\left\|\phi_1^{\text{odd}}\right\|_{L^2(B_1)}}{(2\ell+1)!}\ .$$

We thus obtain the simple bound in terms of the energy

$$\begin{aligned}|a_n^\bullet| &\leq \frac{1}{n!}\,\frac{1}{\sqrt{2}}\,\max\left\{\|\partial_x\phi_0^\bullet\|_{L^2(B_1)},\ \|\phi_1^\bullet\|_{L^2(B_1)}\right\}\\ &\leq \frac{1}{n!}\,\frac{1}{\sqrt{2}}\,\sqrt{\|\partial_x\phi_0^\bullet\|_{L^2(B_1)}^2 + \|\phi_1^\bullet\|_{L^2(B_1)}^2} = \frac{\sqrt{E(\phi^\bullet)}}{n!}\ .\end{aligned}$$

Improved Bound on Coefficients

We decompose the Taylor series into a Taylor polynomial of degree N and the remainder term,

$$\hat{h}^{\bullet}_{\pm} = \hat{h}^{\bullet}_{N} + R^{\bullet}_{N} \qquad \text{with} \qquad \hat{h}^{\bullet}_{N}(\omega) := \sum_{n=0}^{N} a^{\bullet}_{n}\, \omega^{n}\,, \quad R^{\bullet}_{N}(\omega) := \sum_{n=N+1}^{\infty} a^{\bullet}_{n}\, \omega^{n}\,. \tag{17}$$

Improved Bound on Coefficients

Lemma *Let $\mathscr{P}(\omega)$ be a real polynomial of degree at most N with $N \in \mathbb{N}_0$,*

$$\mathscr{P}(\omega) = a_0 + a_1\,\omega + \cdots + a_N\,\omega^N\,.$$

Then for any $\omega_1 > 0$, the highest coefficient of $\mathscr{P}$ satisfies the following inequalities:

$$|a_N| \le \frac{1}{\sqrt{\omega_1}}\sqrt{\frac{2}{\pi}}\left(\frac{4}{\omega_1}\right)^N \|\mathscr{P}\|_{L^2([0,\omega_1])}\left(1+\mathscr{O}\left(\frac{1}{N}\right)\right) \tag{18}$$

$$\le \frac{1}{\sqrt{\omega_1}}\left(\frac{4}{\omega_1}\right)^N \|\mathscr{P}\|_{L^2([0,\omega_1])}\,. \tag{19}$$

Bounding the Remainder

Lemma *Given* $\varepsilon \in [0,1]$ *and* $N \in \mathbb{N}_0$*, we choose*

$$\omega_1 = \left(\varepsilon^2 \, (N+1)!^2 \, (2N+3)\right)^{\frac{1}{2N+3}} . \tag{20}$$

Then the remainder term is bounded on $[0, \omega_1]$ *by*

$$\|R_N^\bullet(\omega)\|_{L^2([0,\omega_1])} \leq 4\varepsilon \sqrt{E(\phi^\bullet)} \, .$$

Proof

We can estimate the remainder by

$$
\begin{aligned}
|R_N^\bullet(\omega)| &\le \sum_{n=N+1}^{\infty} \frac{\omega^n}{n!} \sqrt{E(\phi^\bullet)} \\
&= \frac{\omega^{N+1}}{(N+1)!} \left(1 + \frac{\omega}{N+2} + \frac{\omega^2}{(N+2)(N+3)} + \cdots \right) \sqrt{E(\phi^\bullet)} \\
&\le c(\omega) \frac{\omega^{N+1}}{(N+1)!} \sqrt{E(\phi^\bullet)} \qquad \text{with} \qquad c(\omega) := \sum_{n=0}^{\infty} \left(\frac{\omega}{N+2}\right)^n .
\end{aligned}
\tag{21}
$$

Using this pointwise bound, the L^2-norm can be estimated by

$$
\|R_N^\bullet(\omega)\|^2_{L^2([0,\omega_1]} \le 16E(\phi^\bullet) \int_0^{\omega_1} \frac{\omega^{2N+2}}{(N+1)!^2} d\omega \le \frac{16\, E(\phi^\bullet)}{(N+1)!^2\,(2N+3)} \omega_1^{2N+3}.
$$

Improved Coefficient Bound

Proposition *Assume that*

$$E(\phi_-^\bullet) \leq \varepsilon^2\, E(\phi^\bullet)\,.$$

Then the Taylor coefficients are bounded for all $n \in \mathbb{N}_0$ by

$$|a_n^\bullet| \leq \frac{6}{\sqrt{2n+1}}\,\frac{4^n}{n!}\,\varepsilon^{\frac{2}{2n+3}}\,\sqrt{E(\phi^\bullet)}\,.$$

Proof

$$
\begin{aligned}
\|\hat{h}^{\bullet}_{N}(\omega)\|_{L^2([0,\omega_1])} &= \left\|\hat{h}^{\bullet}_{\pm} - R^{\bullet}_{N}\right\|_{L^2([0,\omega_1])} \le \left\|\hat{h}^{\bullet}_{\pm}\right\|_{L^2([0,\omega_1])} + \left\|R^{\bullet}_{N}\right\|_{L^2([0,\omega_1])} \\
&\le \left\|\hat{h}^{\bullet}_{\pm}\right\|_{L^2([0,\infty))} + \left\|R^{\bullet}_{N}\right\|_{L^2([0,\omega_1])} \le \sqrt{\pi\, E(\phi^{\bullet}_{-})} + \left\|R^{\bullet}_{N}\right\|_{L^2([0,\omega_1])} \\
&\le \varepsilon\sqrt{\pi\, E(\phi^{\bullet})} + 4\varepsilon\sqrt{E(\phi^{\bullet})} \le 6\varepsilon\sqrt{E(\phi^{\bullet})}\,.
\end{aligned}
$$

Applying the previous Lemma to the polynomial $\hat{h}^{\bullet}_{N}$ gives the bound

$$
\begin{aligned}
|a^{\bullet}_{N}| &\le \frac{1}{\sqrt{\omega_1}}\left(\frac{4}{\omega_1}\right)^{N} 6\varepsilon\sqrt{E(\phi^{\bullet})} \\
&= \varepsilon^{\frac{2}{2N+3}}\, 4^{N}\,(N+1)!^{-\frac{2N+1}{2N+3}}\,(2N+3)^{-\frac{2N+1}{4N+6}}\, 6\sqrt{E(\phi^{\bullet})}\,.
\end{aligned}
$$

Coefficient wise estimate

Proposition (F.Finster – C.F.P.) *Assume that the energy of the positive frequency component is bounded in terms of the total energy by*

$$E(\phi_+) < \varepsilon\, E(\phi)\,.$$

Then the even and odd components of the initial data in momentum space are bounded pointwise for all $k \in \mathbb{R}$ by

$$\left|\hat{h}(k)\right| \leq \left|k\,\hat{\phi}_0(k)\right| + \left|\hat{\phi}_1(k)\right| \leq 12\,\sqrt{E(\phi)}\,|4k|^{-\frac{3}{2}}\,g\big(4\,|k|\big)\,, \tag{22}$$

where g is the series

$$g(\omega) := \sum_{n=0}^{\infty} \frac{1}{\sqrt{2n+1}}\,\frac{(4\omega)^{n+\frac{3}{2}}}{n!}\,\varepsilon^{\frac{2}{2n+3}}\,. \tag{23}$$

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$$\left|\hat{h}(k)\right| \leq \left|k \, \hat{\phi}_0(k)\right| + \left|\hat{\phi}_1(k)\right| \leq 6^{\frac{3}{2}} \, \frac{\sqrt{E(\phi)}}{\sqrt{2e|\log(\varepsilon)|}} e^{4\omega}. \tag{24}$$

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Improvement over constant bound as long as

$$\frac{6^{\frac{3}{2}}\, e^{4\omega}}{\sqrt{4e|\log(\varepsilon)|}} \leq 1. \tag{25}$$

Proof

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{2n+1}} \frac{(4\omega)^n}{n!} \varepsilon^{\frac{2}{2n+3}} \leq \sqrt{\frac{3}{2}} \sum_{n=0}^{\infty} \sqrt{\frac{2}{2n+3}} \frac{(4\omega)^n}{n!} \varepsilon^{\frac{2}{2n+3}}$$
$$\leq \sqrt{\frac{3}{2}} \max_{n\in[0,\infty)} \left[\sqrt{\frac{2}{2n+3}} \varepsilon^{\frac{2}{2n+3}}\right] \sum_{n=0}^{\infty} \frac{(4\omega)^n}{n!} \leq \sqrt{\frac{3}{2}} \sup_{x\in\mathbb{R}^+} \left[x\, e^{x^2 \log \varepsilon}\right] e^{4\omega},$$

where in the last step we set $x = \sqrt{2/(2n+3)}$. In order to estimate the last supremum, we set $y = \sqrt{-\log \varepsilon} x$,

$$\sup_{x\in\mathbb{R}^+} \left[x\, e^{x^2 \log \varepsilon}\right] = \frac{1}{\sqrt{-\log \varepsilon}} \sup_{y\in\mathbb{R}^+} y\, e^{-y^2} = \frac{1}{\sqrt{2e\,|\log \varepsilon|}}$$

L^2 Weight

It follows immediately that

$$\left\|\hat{h}(k)\right\|_{L^1_{[0,\omega_{max}(\varepsilon)]}} < \frac{1}{4}\sqrt{E(\phi)} \quad \text{and} \quad \left\|\hat{h}(k)\right\|^2_{L^2_{[0,\omega_{max}(\varepsilon)]}} < \frac{1}{8}E(\phi). \tag{26}$$

By Plancherel we know that the L^2 norm of a function is conserved under Fourier transform and hence we know that

$$E(\phi) \geq \left\|\hat{h}(k)\right\|^2_{L^2_{[\omega_{max}(\varepsilon),\infty)}} > \frac{7}{8}E(\phi). \tag{27}$$

It is clear that $\omega_{max}(\varepsilon)$ is monotone decreasing in $\varepsilon \in (0,1]$ with $\lim_{\varepsilon\to 0}\omega_{max}(\varepsilon) = \infty$

Best Estimate

It turns out that we can write g as a solution to a Goursat problem and obtain the following bound

Proposition (F.Finster – C.F.P.) $g(a,b)$ *is bounded by*

$$\begin{aligned} |g(a,b)| &\lesssim e^{3a}\, \exp\left(\frac{3}{2}\,\mathrm{Im}^2\, y_0 + \sqrt{2b}\left(\frac{1}{2\,\mathrm{Im}\, y_0} - \mathrm{Im}\, y_0\right)\right)\sqrt{\frac{e^{-\nu}}{\sqrt{\nu}}\,\mathrm{Erfi}(\nu)} \\ &= e^{\frac{5a}{2}}\, \exp\left(\frac{5}{4}\,\mathrm{Im}^2\, y_0 + \sqrt{2b}\left(\frac{1}{2\,\mathrm{Im}\, y_0} - \mathrm{Im}\, y_0\right)\right)\sqrt{e^{-\nu}\,\mathrm{Erfi}(\nu)}\,, \end{aligned}$$

where $\mathrm{Im}\, y_0$ *and* ν *are given by*

$$\sqrt{2b} = 3\,\mathrm{Im}\, y_0 + 2e^{2a}\,\mathrm{Im}\, y_0\, e^{\mathrm{Im}^2\, y_0} \qquad (28)$$

$$\nu = e^{2a}\, e^{\mathrm{Im}^2\, y_0}\,. \qquad (29)$$

where $a(\varepsilon)$ and $b(\omega)$

Sketch of Proof

Differentiating the function $g(a,b)$ with respect to a and b gives

$$\partial_a g(a,b) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{1}{\sqrt{2n+1}}\,(2n+3)\,e^{(2n+3)\,a-\frac{b}{2n+3}}$$

$$\partial_b \partial_a g(a,b) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{1}{\sqrt{2n+1}}\,\Big(-\frac{2n+3}{2n+3}\Big)\,e^{(2n+3)\,a-\frac{b}{2n+3}} = -g(a,b)\,.$$

Hence g is a solution of the PDE

$$(\partial_a \partial_b + 1)\,g = 0\,. \tag{30}$$

Goursat Problem

Introducing the coordinates

$$T = a + b\,, \qquad X = a - b$$
$$\partial_T = \frac{1}{2}\big(\partial_a + \partial_b\big)\,, \qquad \partial_X = \frac{1}{2}\big(\partial_a - \partial_b\big)\,,$$

the equation takes the more familiar form

$$\left(\partial_T^2 - \partial_X^2 + 1\right) g = 0\,.$$

This PDE comes with initial conditions at $b = 0$ given by the series

$$g_0(a) := g(a, 0) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{1}{\sqrt{2n+1}}\,e^{(2n+3)\,a}\,. \tag{31}$$

Moreover, Lebesgue's monotone convergence theorem implies that

$$\lim_{b\to\infty} g(a, b) = \lim_{a\to-\infty} g(a, b) = 0\,. \tag{32}$$

3+1 Dimensions

Assume that for $\varepsilon \in (0, 1]$, the energy of the negative-frequency component is bounded in terms of the total energy by

$$E(\phi_-) \leq \varepsilon^2 \, E(\phi) \, .$$

Then the L^2-norm of the spatial Fourier transform on a sphere of radius ω is bounded for all $\omega \in \mathbb{R}^+$ by

$$\int_{S^2} \left|\omega \, \hat{\phi}(\vartheta, \phi, \omega)\right|^2 d\mu_S^2(\vartheta, \varphi) \leq 625 \, d_0^{\frac{10}{3}} \; C \; E(\phi) \; \left(4\omega\right)^{-\frac{6}{2}} \, g_0^2\left(\omega, \varepsilon\right) \, ,$$

where C is the constant

$$C := \sum_{l=0}^{\infty} (2l+1) \, d_l^{\frac{4l+6}{2l+5}} < \infty \tag{33}$$

(and the d_l are given by $d_l := \frac{4\pi}{\sqrt{6\,(2l+1)}} \, \frac{l!}{(2l-1)!!}$.)

Thanks

Thank you for your attention.